HYDROSTATICS AND PNEUMATICS


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HYDROSTATICS AND PNEUMATICS

(THE MECHANICS OF FLUIDS).

BY

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Aqua est labile corpus naturaliter.—Dante.

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PREFACE.

The design of this elementary text-book is to give an account of the fundamental principles of Hydrostatics and Pneumatics, and to show how the subject may be developed by simple mathematical methods.

To meet the case of students who begin the study of the Mechanics of Fluids with not more than an elementary knowledge of the Mechanics of Solids, chapters on Units and on the Principles of Statics precede the consideration of the general problems of fluid pressure. For the same reason short discussions of Uniform Circular Motion and of Harmonic Motion are also included in the book.

The method of integration has been explained without the use of the notation of the *Integral Calculus*, and has been applied to the solution of such problems as those of finding moments of inertia, centres of pressure, work done by a variable force, &c.

At the end of each chapter there is given a collection of examples which are left as exercises for the student. In some cases these examples have been divided into two sets, marked A and B respectively, the examples in the set marked B being of a higher order of difficulty than those in the set marked A. A few of the examples are original, and the remainder have been taken from examination papers set in the South Kensington, Civil Service, and University Examinations. A collection of miscellaneous examples and specimen examination papers are added at the end.

I have to acknowledge my indebtedness to Deschanel's *Natural Philosophy*, from which I have borrowed, with the concurrence of the publishers, illustrations of several hydrostatic and pneumatic instruments and machines.

It is hoped that the book will be found suitable for students of Science Classes, and for others who are preparing for elementary examinations in Hydrostatics and Pneumatics.

R. H. PINKERTON.

October, 1893.

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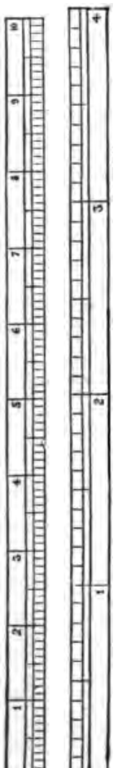
FRENCH AND ENGLISH MEASURES.

A DECIMETRE DIVIDED INTO CENTIMETRES AND MILLIMETRES.

INCHES AND TENTHS.



1 millimetre = ·03937 inch, or about $\frac{1}{25}$ inch.
1 centimetre = ·3937 inch.
1 decimetre = 3·937 inches.
1 metre = 39·37 inches = 3·281 feet.

1 sq. centimetre = ·155 sq. in.

1 cub. centimetre = ·061025 cub. in.

1 gramme = 15·432 grain.
1 kilogramme = 2·205 lbs. avoir.

1 gramme per sq. centim. = 2·0481 lbs. per sq. ft.

1 kilogrammetre = 7·2331 foot-pounds.

1 inch = 2·54 centimetres.
1 foot = 30·48 centimetres.

1 sq. inch = 6·45 sq. centimetres.
1 sq. foot = 929 sq. centimetres.

1 cub. inch = 16·39 cub. centimetres.
1 cub. foot = 28316 cub. centimetres.

1 grain = ·0648 gramme.
1 oz. avoir. = 28·35 grammes.
1 lb. avoir. = 453·6 grammes.

TABLE OF RELATIVE DENSITIES.

Liquids.

Pure water at 4° C.,	1·000
Sea water, ordinary,	1·026
Alcohol, pure,	·791
„ proof spirit,	·916
Ether,	·716
Mercury at 0° C.,	13·596

Solids.

Brass,	7·8 to 8·6
Copper,	8·6 to 8·9
Gold,	19 to 19·6
Iron, cast,	6·9 to 7·3
„ wrought,	7·6 to 7·8
Lead,	11·4
Platinum,	21 to 22
Silver,	10·5
Steel,	7·8 to 7·9
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Ice,	·92
Basalt,	3·0
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Chalk,	1·8 to 2·8
Clay,	1·9
Glass, crown,	2·5
„ flint,	3·0
Quartz (rock crystal),	2·65
Sand,	1·4
Fir, spruce,	·48 to ·7
Oak, European,	·69 to ·99
Lignum vitæ,	·65 to 1·33
Sulphur, octahedral,	2·05
„ prismatic,	1·98

GASES, at 0° C., and a pressure of 76 centimetres of mercury (one atmosphere).

Air, dry,	·0012932
Oxygen,	·0014298
Nitrogen,	·0012562
Hydrogen,	·00008957
Carbonic acid,	·0019774

HYDROSTATICS AND PNEUMATICS.

CHAPTER I.—INTRODUCTORY.

1. Body.—Mass.—Particle.—Force.

A *body* is a limited portion of matter.

The quantity of matter in a body is called the *mass* of the body.

A *particle* is an infinitesimal portion of matter. Thus a body of finite size may be conceived to be made up of an infinite number of particles.

Force is any cause which produces, or tends to produce, change of motion in a body.

Bodies are bounded by surfaces, which may be flat or curved; and surfaces are bounded by lines, which may be straight or curved. In Mensuration, which is a branch of Geometry, formulæ are investigated for finding the volumes of bodies and the areas of surfaces whose dimensions are known. All the formulæ in Mensuration which will be required in this book are given in Art. 14, page 17.

2. The Three States of Matter.

Matter exists in three states:—(1) the **solid state**, (2) the **liquid state**, (3) the **gaseous state**.

Liquids and gases are classed together under the name of **fluids**.

3. Divisions of the Science of Dynamics.

Dynamics is that branch of Natural Philosophy which treats of the action of force upon matter, and is divided into two parts—**Statics** and **Kinetics**.

Statics treats of the relations between forces which maintain bodies in a state of rest. Kinetics investigates the action of force in causing or changing motion. The Science of Motion, treated apart from the consideration of force, is called **Kinematics**.

That part of Statics in which the bodies under consideration are fluids is called **Hydrostatics**. This term is sometimes used in a more restricted sense as the Statics of Liquids, the equilibrium of gases being discussed under **Pneumatics**.

This is the newer nomenclature. In the older nomenclature, to which some writers still adhere, the Science of Force was called Mechanics, and the term Dynamics was applied to that part of the subject which is now commonly called Kinetics.

4. Solids and Fluids.

Everyone is familiar with the distinguishing properties of solid and fluid bodies. *Solid* bodies maintain, under varying circumstances, a permanent form, and resist the action of forces tending to change their form. This permanence of form is due to molecular forces, called *forces of cohesion*, which resist the action of external forces tending to change the relative positions of the particles of the body. Thus a solid body, such as a rod of iron, resists the action of forces tending either to stretch it, to compress it, to twist it, or to bend it.

On the other hand, we associate the term *fluid* with bodies which possess the property, well marked in water, alcohol, and air, of yielding to the action of the smallest force tending to change their shape. A fluid therefore has no permanent form, and we may take this as the property on which to base our **definition of a fluid:—**

Def.—*A fluid is a body which yields to the action of the smallest force tending to change its shape.*

It follows that a solid can sustain and transmit a force, whether it be a tension or a pressure, and that a fluid cannot sustain nor transmit a tension. The investigation of the laws under which fluids, when contained in vessels, can sustain, transmit, and exert pressure is the main problem of Hydrostatics.

It is usual in elementary dynamics to treat solids as *perfectly rigid* bodies, that is, as bodies which are absolutely unchanged in shape by the forces which act on them. No solid in nature is perfectly rigid, so that some of the results of Abstract Dynamics must be modified before they can be applied to the solution of problems in practical mechanics. The consideration of the corrections which must be applied to the results of

Abstract Dynamics for imperfect rigidity in solid bodies belongs to the Science of Elasticity.

5. Liquids and Gases.

There are important differences between some of the properties of the two classes of fluids.

In the first place, liquids are nearly incompressible, while gases have their volumes sensibly diminished by the application of moderate pressures. The results that are obtained from the principles of Hydrostatics on the supposition that water and other liquids are absolutely incompressible do not differ materially from the results of observation. On the other hand, the compressibility of air, which may be taken as the type of gases, is well known, and is illustrated in the air-gun and in a boy's pop-gun.

Again, gases possess a tendency to expand indefinitely, a property which is not exhibited by liquids. Hence gases, such as coal gas, steam, chlorine, &c., must be kept in closed vessels.

We may therefore give the following **definitions of a liquid** and **of a gas** respectively :—

Def.—*A liquid is a fluid which is nearly incompressible, and which exhibits no tendency to expand.*

Def.—*A gas is a fluid which can be readily compressed, and which possesses the property of tending to expand indefinitely.*

6. Viscosity.—Perfect Fluid.

In some liquids there is observed to be a certain amount of resistance to the sliding of one part over another part, so that those liquids do not immediately change their shape on the application of force. This frictional resistance to the sliding of one particle of a fluid over another particle is called *viscosity*, and fluids which exhibit it in a marked degree are usually called *viscous* fluids. There is no fluid in nature in which there is absolutely no viscosity. The viscosity of such liquids as tar, syrup, and solution of gum is known to everyone, and the viscosity of more mobile liquids, such as water, may be inferred from many phenomena.

Corresponding to the ideal perfectly rigid solid is the ideal *perfect fluid*, in which there is absolutely no resistance to change

of shape. All fluids in nature exhibit viscosity, or want of perfect fluidity, but in considering the equilibrium of fluids it is not necessary to take viscosity into account. For a viscous fluid under any circumstances will assume, when in equilibrium, precisely the same form as it would assume if it were a perfect fluid. Thus, for example, a quantity of a viscous liquid, such as a solution of gum, when poured into a vessel, will, under the action of gravity, ultimately come to rest with its free surface horizontal. The gum will, in fact, slowly come to rest in the form which would be assumed almost immediately by the same volume of a mobile liquid such as water.

Chapter II.—UNITS.

Principles of Measurement, Arts. 7 to 9.

7. Method of measuring quantities.

To measure a quantity is to find a number which indicates the magnitude of the quantity. Whatever be the kind of quantity we are dealing with, we must first choose some definite **quantity of the same kind** as our standard or **unit of measurement**, and then determine, directly or indirectly, the number of times that the quantity we are measuring contains the unit. The number of times that the quantity contains the unit is the number which represents the quantity, and is called the **numerical value** of the quantity.

In dynamics we have to consider quantities of different kinds —velocities, accelerations, momenta, forces, pressures, quantities of energy, &c.—each of which must be measured in terms of a unit of the same kind. Thus the numerical value of a velocity is the number of times the velocity contains the unit of velocity; the numerical value of a force is the number of times the force contains the unit of force; and so on.

It does not lie within the scope of this book to enter into the question of how the measurement of physical quantities is actually carried out. For descriptions of the pieces of apparatus required in actual measurements,

and for explanations of the method of using them, the student must consult works on Practical Physics. Here we are concerned only with the principles on which the measurements are based.

8. The numerical value of a quantity varies inversely as the magnitude of the unit.

It is at once evident that the numerical value of a given quantity depends on the magnitude of the quantity of the same kind which we select as our unit. If therefore we change our unit, the numerical value of a given quantity will in consequence be changed. Thus if we double our unit, the given quantity will be compared with a quantity twice as large as the original unit, and therefore the number of times that the quantity contains the new unit will be only half the number of times that it contains the old unit. In other words, by doubling the unit we halve the numerical value of a given quantity.

In general, if the unit is altered in the ratio of m to 1, the numerical value of a given quantity will be altered in the ratio of 1 to m. For the new unit is m times the old unit, and therefore the number of times that the given quantity contains the new unit is $(1/m)^{\text{th}}$ of the number of times that it contains the old unit.

Take, as an illustration of this general principle, a familiar case of change of units. Suppose that the numerical value of a certain length is n when the unit of distance is a foot; then the numerical value of the same length when the unit of distance is a yard will be $n/3$. That is, the numerical value is diminished in the ratio 1 to 3 when the unit is increased in the ratio of 3 to 1.

Again, to take a dynamical example, we know that the force which is equal to the weight of one pound of matter at any point of the British Isles is nearly 32 poundals; and therefore the numerical value of a given force, when referred to one poundal as the unit, must be divided by 32 in order to give the numerical value of the same force, when referred to the weight of one pound as the unit. In this case the numerical value of the quantity which is being measured is diminished in the ratio of 1 to 32 when the unit is increased in the ratio of 32 to 1.

9. Fundamental and derived units.

All the quantities we meet with in dynamics are measured in terms of units derived from the units of space, mass, and

time. These units are therefore called the three **fundamental** units of dynamics.

Thus the unit of velocity is the velocity of a point moving over unit distance in unit time; the unit of momentum is the momentum of unit mass moving with unit velocity; the unit of force is the force which produces unit momentum in unit time; the unit of work is the work done by unit force acting through unit distance; and so on.

In this way we construct a system of **derived** units, whose magnitudes depend on the magnitudes of the fundamental units. And it is evident that if one or more of the fundamental units be changed, the derived units will, in general, be changed thereby. Hence it follows that the numerical value of a dynamical quantity depends on the fundamental units.

The advantage of deriving the units of all other quantities from the three fundamental units of space, mass, and time lies in this—that by using such derived units the equations of dynamics are reduced to their simplest forms.

The student must carefully notice that the *equations of dynamics* are not relations between quantities, but *relations between the numerical values of quantities.* It is impossible to compare two quantities of different kinds; we can only compare the numbers which measure them. For example, the statement that a force is equal to the momentum it generates in unit time, if taken literally, is an absurdity; but it expresses a physical fact if it is taken to mean that the number of times that the force contains unit force is equal to the number of times that the momentum generated by the force in unit time contains the unit of momentum.

EXAMPLES.

1. The numerical value of a certain length is m when n feet is the unit of length; show that the numerical value of the same length is n when m feet is the unit of length.

When n feet is the unit of length, the numerical value of the length in question is m, and therefore the number of feet in the length is mn. Hence when m feet is the unit of length, the number of times that the length con-

tains the unit is $mn/m, = n$; that is, the numerical value referred to the new unit is n.

2. How must the formula for uniform velocity, viz. $s = vt$, be modified if the unit of velocity is defined to be the velocity of a body which describes m units of space in n units of time?

If this velocity be taken as the unit, then the velocity v is v times the velocity with which a body describes m units of space is n units of time. Thus with velocity v the body describes mv units of space in n units of time, mv/n units of space in one unit of time, and mvt/n units of space in t units of time. But this space is represented by s· therefore the equation connecting s, v, and t is

$$s = mvt/n.$$

3. A certain length is represented by the number 10 when a furlong is the unit of length. Find the number of inches in that length.

Ans. 79200.

4. A certain angle is represented by the number 3 when an angle of 45° is the unit of angle. Find the number of degrees in the angle.

Ans. 135.

5. Show that the numerical value of a velocity in miles per hour will be found from the numerical value in feet per second by multiplying by the factor 15/22, which may be conveniently called the reducing factor.

6. Determine the reducing factor which will convert miles per hour to yards per minute.

Ans. 88/3.

7. Determine the reducing factor which will convert the numerical value of an acceleration in feet and seconds to its numerical value in miles and hours.

Ans. 27000/11.

8. Show that if the units of space and time are increased m times and n times respectively, the numerical value of a velocity will be multiplied by the factor n/m.

Also show that the numerical value of an acceleration will, by the same change of units, be multiplied by the factor n^2/m.

9. Determine the reducing factor required to convert rainfall in inches to tons per acre.

[Given that one cubic foot of water weighs 1000 oz.]

Ans. 45375/448.

The Two Systems of Units, Arts. 10 to 12.

10. The British and C.G.S. systems of units.

There are two systems of dynamical units in use in this country, the British system and the C.G.S. system, each of which is based on the three units of space, mass, and time.

The fundamental units of the two systems are exhibited in the following table:—

	British System.	C.G.S. System.
Unit of Space,	foot,	centimetre
Unit of Mass,	pound,	gramme.
Unit of Time,	second,	second.

The C.G.S. system—so called from the initial letters of the words centimetre, gramme, second—is the system of units in which physicists now express all experimental results. The British system is still in popular use in this country, but is likely to be ultimately replaced by the C.G.S. system. In our examples and illustrations of the principles of Hydrostatics we shall not restrict ourselves to the exclusive use of either system. In an example it will, of course, be necessary to express in the same system the numerical values of all quantities involved in the example.

11. **The fundamental units.**

The unit of time is the same in both systems, and is the **mean solar second.** It is the second measured by a watch or clock which keeps correct time.

The **foot** is the third part of the yard, which is defined by Act of Parliament to be the distance between two marks made in a certain bar of platinum deposited in the office of the Warden of the Standards in London.

The **pound** is defined in a similar way to be the mass of a certain lump of metal deposited in the same office.

The C.G.S. units of length and mass, the **centimetre** and the **gramme**, are defined with reference to standards, the *metre* and the *kilogramme* respectively, which are deposited in the Government Offices at Paris. The centimetre is the hundredth part of the metre, and the gramme is the thousandth part of the kilogramme. The metre is equal to 39·37 inches approximately, so that the centimetre is nearly ·3937 of an inch, that is, about 4/10 of an inch. The kilogramme is nearly $2\frac{1}{5}$ pounds, so that the gramme is about $15\frac{1}{2}$ grains.

12. Advantages of the metric system o. measures.

The C.G.S. units of length and mass are selected from the

French or metric system of measures, which possesses certain advantages over the British system.

In the metric system all measures of length are obtained from the metre by multiplying or dividing by a power of 10, and all measures of mass are derived in a similar way from the kilogramme.

Thus the metre (abbreviation, m.) is divided into 10 decimetres (dcm.), the decimetre into 10 centimetres (cm.), and the centimetre into 10 millimetres (mm.).

Hence 1 m. = 10 dcm. = 100 cm. = 1000 mm.
1 dcm. = 10 cm. = 100 mm.
1 cm. = 10 mm.

The prefixes *deci-*, *centi-*, *milli-* in the submultiples of the metre therefore denote 1/10, 1/100, 1/1000 respectively. These prefixes are derived from the *Latin*.

The prefixes *deca-*, *hecto-*, *kilo-* are used to denote multiplication by 10, 100, 1000 respectively. These prefixes are derived from the *Greek*. Thus a decametre is 10 metres, a hectometre is 100 metres, and a kilometre is 1000 metres. These names of the multiples of the metre are not, however, so often required in dynamics as the names of the submultiples.

The prefixes are used with the same meanings in expressing multiples and submultiples of the gramme. Thus a kilogramme is 1000 grammes, a decigramme is 1/10 of a gramme, a centigramme is 1/100 of a gramme; and so on.

The convenience of this system of multiples and submultiples is evident, as a change may be made from one measure to another by multiplication or division by a power of 10; and a number may be multiplied or divided by a power of 10 by merely changing the position of the decimal point.

On page v. a table is given showing the relations between English and French measures.

Derived Units, Arts. 13 to 19.

13. Derived geometrical units.—Units of Area and Volume.

The unit of area is the area of a square whose side is the unit of length.

The unit of volume is the volume of a cube whose edge is the unit of length.

In the British system the unit of area is the area of a square whose side is one foot. For most purposes it is convenient in Hydrostatics to use the square inch, instead of the square foot, as the unit of area. The square foot contains $12^2, = 144$ square inches. The unit of volume in the British system is the cubic foot, but it is sometimes convenient to use, instead of the cubic foot, the cubic inch. The cubic foot contains $12^3, = 1728$ cubic inches.

In the C.G.S. system the unit of area is the square centimetre, that is, the area of a square whose side is one centimetre. The unit of volume is the cubic centimetre, that is, the volume of a cube whose edge is one centimetre. The *litre* is the name given to a cubic decimetre, so that the litre is 1000 cubic centimetres.

14. Mensuration of surfaces and volumes.

The tables on page 17 contain the formulæ by which the areas and volumes of certain surfaces and solids are calculated when the dimensions are known. The letter π, which frequently occurs in the tables, is the symbol which is always used to represent the ratio of the circumference of a circle to its diameter. This ratio is approximately equal to 22/7, and more nearly equal to 3·1416. In all numerical calculations in this book π will be taken equal to 22/7.

EXAMPLES I.

[Take $\pi = 22/7$.]

(*The Answers are given on page* 332.)

1. Give the number of centimetres in each of the following lengths :—

(i) 10 decimetres. (ii) 37 metres. (iii) 3·87 kilometres. (iv) 2·3 millimetres. (v) 6·32 decametres.

2. Express the following masses in grammes :—

(i) 3·2 kilogrammes. (ii) 450 decigrammes. (iii) 3·73 milligrammes.

3. State the number of square centimetres in the areas of the squares whose sides are—(i) 1 metre. (ii) 3·4 decimetres. (iii) ·003 kilometre. (iv) ·23 decimetre.

4. Find the number of cubic centimetres in the cubes whose edges are—(i) $1\frac{1}{2}$ metres. (ii) 3·8 millimetres. (iii) ·23 decimetre.

5. Find in square feet the areas of the following figures:—(i) A rectangle whose adjacent sides are 3 ft. 4 in. and 5 ft. 6 in. respectively.

I.—LENGTH.

Circumference of a circle, radius r, $= 2\,\pi\,r$.

II.—TABLE OF AREAS.

Figure.	Dimensions.	Formula for Area.
Square,	side a,	a^2
Rectangle,	adjacent sides a, b,	$a\,b$
Parallelogram,	base a, altitude h,	$a\,h$
Triangle,	base a, altitude h	$\frac{1}{2}\,a\,h$
Circle,	radius r,	$\pi\,r^2$
Surface of sphere,	radius of sphere r,	$4\,\pi\,r^2$
Curved surface of cylinder,	radius of base r, length of axis h,	$2\,\pi\,r\,h$
Curved surface of cone,	radius of base r, length of axis h,	$\pi\,r\,\sqrt{r^2+h^2}$

III.—TABLE OF VOLUMES.

Figure	Dimensions.	Formula for Volume.
Cube,	edge a,	a^3
Rectangular body,	edges a, b, c,	$a\,b\,c$
Cylinder,	radius of base r, length of axis h,	$\pi\,r^2\,h$
Cone,	radius of base r, length of axis h,	$\frac{1}{3}\,\pi\,r^2\,h$
Prism,	area of base A, altitude h,	$A\,h$
Sphere,	radius r,	$\frac{4}{3}\,\pi\,r^3$

(ii) A circle whose radius is 2 ft. (iii) The curved surface of a cylinder whose radius is 1 ft. 6 in. and height 6 ft. (iv) The six faces of a rectangular solid whose edges are 2 ft., 3 ft., and 4 ft. respectively.

6. Find the number of cubic inches in each of the following bodies:— (i) A cube whose edge is 4 in. (ii) A rectangular body whose edges are 1 ft. 4 in., 4 in., and 3 in. respectively. (iii) A cylinder whose height is 6 in., and base a circle of 6 in. diameter. (iv) A cone whose height is 1 ft. 6 in., and base a circle of 10 in. diameter. (v) A sphere whose radius is 4 in.

7. A circular hole of 3 inches radius is cut out of a circular plate of 10 inches radius. Find the area of the remainder of the plate in square inches.

8. In a wooden sphere, whose radius is 5 inches, there is a spherical cavity 4 inches in diameter. Find the volume of wood in the sphere.

15. Dynamical units of force and work.

The units of force and work derived from the fundamental units are called **absolute** or **dynamical units**, and must be distinguished from **gravitational units** of force and work, which have reference to the force of gravity.

In the British system the unit of velocity is the velocity of 1 ft./sec., and the unit of momentum is the momentum of one pound moving with the velocity of 1 ft./sec. Hence we have as the definition of the **British absolute unit of force** the following:—

Def.—*The absolute unit of force in the British system of units is the force which acting on one pound for one second produces in that mass a velocity of one foot per second.*

This force is called the **Poundal.**

Again the **British absolute unit of work** is defined as follows:—

Def.—*The absolute unit of work in the British system of units is the work done by a poundal acting through the distance of a foot.*

This quantity of work is called the **Foot-poundal.**

In the C.G.S. system the absolute unit of force is called the **dyne**, and the absolute unit of work is called the **erg**. The dyne and the erg are defined as follows:—

Def.—*The absolute unit of force in the C.G.S. system, the dyne, is the force which acting on one gramme for one second produces in that mass a velocity of one centimetre per second.*

Def.—*The absolute unit of work in the C.G.S. system, the erg, is*

the work done by the dyne acting through the distance of one centimetre.

16. Equation of momentum for rectilinear motion.

If a mass m is moving in a straight line under the action of a constant force of P absolute units, and if in t units of time the velocity increases from V to v, then

$$P t = m v - m V,$$

an equation which we shall refer to as the **equation of momentum.**

We know from the Second Law of Motion that a force is proportional to the change of momentum it produces in unit time, and that a constant force, acting for any time, produces the same change of momentum in each unit of time. In this case the whole change of momentum in t units of time is

$$m v - m V,$$

and therefore the change of momentum in one unit of time is

$$(m v - m V)/t.$$

Hence the force P bears to the absolute unit of force the ratio of $(m v - m V)/t$ to 1.

Hence the number of times that P contains the absolute unit of force is $(m v - m V)/t$.

Therefore $\qquad P = (m v - m V)/t,$

or $\qquad P t = m v - m V.$

If British units are used, P will be expressed in poundals, t in seconds, m in pounds, v and V in feet per second.

If C.G.S. units are used, P will be expressed in dynes, t in seconds, m in grammes, v and V in centimetres per second.

Ex. 1.—There are two bodies whose masses are in the ratio of 2 to 3, and their velocities are in the ratio of 21 to 16. What is the ratio of their momenta?

If their momenta are due to forces P and Q acting on the bodies respectively for equal times, what is the ratio of P to Q?

The momenta are expressed in terms of the same unit of momentum by the numbers 2×21 and 3×16. Hence their momenta are in the ratio of

$$2 \times 21 : 3 \times 16, \text{ or } 7 : 8.$$

If t represent the time in which their momenta are generated by the forces P and Q,

then $$P\,t : Q\,t = 7 : 8,$$
or $$P : Q = 7 : 8.$$

Ex. 2.—A mass of 10 kilogrammes is moving at a certain instant with a velocity of 15 cm./sec. From that instant a constant force acts on the mass in the direction opposite to that of the motion of the mass, and due to the action of this force the mass is brought to rest in 100 seconds. Find the force in dynes.

If P denote the number of dynes in the force, then the momentum *destroyed* by the force P in 100 seconds is 100 P C.G.S. units of momentum. But this is equal to the initial momentum of the mass, which is $10 \times 1000 \times 15$ C.G.S. units of momentum.

Hence $$100\,P = 10 \times 1000 \times 15,$$
or $$P = 1500 \text{ dynes}.$$

17. Energy.

Def.—*The energy of a body is the work which the body can do in virtue of its motion or of its position.*

That work can be done by a body in virtue of its motion may be illustrated by the case of a shot fired from a heavy gun. The shot, in virtue of its motion, may penetrate several inches of iron plate, that is, the shot may do work in overcoming through as many inches the resistance of the plate to penetration.

The work which a body can do in virtue of its velocity is called its **Kinetic Energy.**

Again, work may be obtained from a body by allowing it to fall from any level to a lower level. This is illustrated in the case of the pile-driver, in which the ram, a massive piece of iron, is raised to a height of several feet above the top of the pile which is to be driven into the ground. In this position the ram possesses a store of energy due to its height above the pile. On being released, it falls under the action of gravity, its energy of position being transformed, as it falls, into energy of motion. At the moment when it strikes the pile its energy of position has all been transformed into kinetic energy, which is expended during the blow in driving the pile a certain distance into the ground against the resistance of the ground to penetration.

The work which may be obtained from a body in virtue of its position is called **Potential Energy.**

18. **Equation of energy for rectilinear motion.** If a mass m is moving in a straight line under the action of a constant force of P absolute units, and if the velocity increases from V to v while the body passes over the distance s, then

$$Ps = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mV^2, \qquad (1)$$

an equation which we shall refer to as the **equation of energy.**

Let a be the constant acceleration which the force P produces in the mass m, then, by the Second Law of Motion,

$$P = ma.$$

But by the formula for uniformly accelerated motion

$$v^2 = V^2 + 2as.$$

Multiply both sides of the last-written equation by $m/2$, and then replace ma by P. The result is

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mV^2 + Ps,$$

or

$$Ps = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mV^2,$$

the required equation.

If the body starts from rest, then $V = 0$, and the equation becomes

$$Ps = \tfrac{1}{2}mv^2. \qquad (2)$$

If the force acts in the direction opposite to that of the initial motion, then the sign of P in formula (1) must be changed. Hence for this case we have, corresponding to the equation (1), the equation

$$Ps = \tfrac{1}{2}mV^2 - \tfrac{1}{2}mv^2, \qquad (3)$$

where V is the initial velocity, and v the velocity when the space s has been described. In this case the body is moving against the action of the force P, which is therefore called a *resistance*, and the velocity of the body will decrease, so that v will be less than V.

When the body has been brought to rest by the action of the resistance P, then $v = 0$, and the equation (3) then becomes

$$Ps = \tfrac{1}{2}mV^2. \qquad (4)$$

The equations (1), (2), (3), and (4) may be expressed in words by using the terms work and energy.

In equation (4), the term on the right-hand side, Ps, represents in absolute units the work which the body does against the resistance P before it is brought to rest. The equation shows that this amount of work is equal to $\frac{1}{2}mV^2$. Hence $\frac{1}{2}mV^2$ is the formula for the work which a body, mass m, moving with the velocity V, can do in virtue of its motion. Thus—

I. *$\frac{1}{2}mV^2$ is the formula for the kinetic energy of a mass m moving with the velocity V.*

In equations (1) and (2), Ps is the work done *on* the body by the force P while the body moves over the distance s, and the equations show that—

II. *When a force does work on a body, the body gains kinetic energy equal to the amount of work done.*

Again, in equation (3), Ps represents the work done *by* the body against the resistance P while the body moves over the distance s, and the equation shows that—

III. *When a moving body does work against a resistance, the body loses kinetic energy equal to the amount of work done.*

The student must carefully notice that in the equations of this Article force and work must be expressed in absolute units. Thus if we are using British units, masses will be expressed in pounds, distances in feet, velocities in feet per second, forces in poundals, and amounts of work in foot-poundals. If C.G.S. units are being used, masses will be expressed in grammes, distances in centimetres, velocities in centimetres per second, force in dynes, and amounts of work in ergs.

Ex. 1.—A body, whose mass is 10 lbs., is acted on by a force of 9 poundals. Find the space passed over by the body while its velocity increases from 8 ft./sec. to 17 ft./sec.

Here we use equation (1). We are given that

$$P=9,\ m=10,\ v=17,\ V=8,$$

and it is required to find s. The equation gives

$$9s=\tfrac{1}{2}\times 10\times 17^2-\tfrac{1}{2}\times 10\times 8^2,$$
$$=1125;$$

therefore

$$s=125 \text{ feet, the distance required.}$$

Ex. 2.—A mass of 20 lbs. is moving at a certain instant with a velocity of 21 ft./sec. From that instant it moves against a constant resistance of 30 poundals. Find its velocity when it has moved over a distance of 39 feet.

Here we use equation (3). We are given that

$$P=30,\ s=39,\ m=20,\ V=21,$$

and it is required to find v. The equation gives

$$30\times 39=\tfrac{1}{2}\times 20\times 21^2-\tfrac{1}{2}\times 20\,v^2,$$

or

$$10v^2=4410-1170,\ =3240.$$

Hence

$$v^2=324, \text{ and } v=18 \text{ ft./sec.}$$

Ex. 3.—A body, whose mass is 100 grammes, and which is initially at rest, has 125000 ergs of work done on it. What is then the velocity of the body?

The kinetic energy of the body is equal to the work done on it. Hence if v be the velocity required, we have

$$\tfrac{1}{2}\times 100\times v^2=125000$$

or

$$v^2=2500;$$

therefore

$$v=50 \text{ cm./sec.}$$

19. Measurement of pressure in fluids.

When two solid bodies, A and B say, are pressed together, the body A exerts a force on the body B, and the body B acts on the body A with an equal and opposite force. This mutual action is called the *stress* between the two bodies, the force which A exerts on B and the force which B exerts on A being the two *aspects* of the stress. These forces are called in the dynamics of solids the pressure of A on B and the pressure of B on A respectively.

In hydrostatics the term **pressure** is used to denote **force per unit of area,** and it is in this sense that the term will be used in this book. The expression *intensity of pressure* is sometimes used with the same meaning.

The terms *total pressure* and *resultant pressure* are also used in hydrostatics. The meanings of these terms will be explained in a later chapter.

The absolute unit of hydrostatic pressure in the British system of units is the pressure of one poundal per square foot.

The absolute unit of hydrostatic pressure in the C.G.S.

system of units is the pressure of one dyne per square centimetre.

EXAMPLES II.

(*The Answers are given on page* 332.)

1. A constant force, acting on a body whose mass is 12 lbs. for 10 seconds, gives the body a velocity of 15 ft./sec. Express the force in poundals, and find the velocity it would give in 5 seconds to a mass of 8 lbs.

2. A force of 32 poundals acts on a body, initially at rest, for $3\frac{1}{2}$ seconds. Find the momentum of the body at the end of that time.

3. A force of 125 poundals acts on a mass of 18 oz. for $3\frac{1}{2}$ minutes. Find the velocity of the mass at the end of that time.

4. A body, acted on by a constant force of $16\frac{2}{3}$ poundals, has its velocity increased from 100 ft./sec. to 150 ft./sec. during an interval of one minute. What is the mass of the body?

5. A mass of 20 lbs., moving with a velocity of 40 feet per second, is brought to rest by a constant force, acting in the direction opposite to the direction of motion, after passing over 400 feet. What is the force in poundals?

6. A constant force, acting on a mass of 20 grammes, gives the mass a velocity of 24 cm./sec. in 3 seconds. Express the force in dynes.

7. Find the kinetic energy of a mass of 150 lbs. moving with the velocity of 180 ft./sec.

If this kinetic energy were communicated to a free stationary mass of 200 lbs., with what velocity would that mass move?

8. What constant force would be required to stop a mass of 100 grammes moving with the velocity of 100 cm./sec. (i) in 100 seconds, (ii) in 100 centimetres?

9. What constant horizontal force, expressed in poundals, is required to stop a train of 100 tons mass moving at 50 miles an hour (i) in one minute, (ii) in 200 yards?

10. A mass of 81 lbs., moving with an initial velocity of 200 feet per second, is acted on by a uniform retarding force of 108 poundals. How far will the mass move while its velocity falls from 200 to 196 ft./sec.? Also, while it falls from 4 ft./sec. to zero?

11. A body, whose mass is 120 grammes, moving with a velocity of 40 cm./sec., is brought to rest by a constant resistance after passing over 4000 centimetres. Find the resistance in dynes.

12. A square, whose side is 4 feet, is subject to a uniform pressure of 3000 poundals per square foot. Find the whole force exerted on the area of the square.

13. The whole force on a circle, radius 50 centimetres, is 35000 dynes. Assuming that the pressure is constant over the area, find the pressure in dynes on each square centimetre.

[Take $\pi = 22/7$.]

Gravitational Units, Arts. 20 and 21.

20. Gravitational units.

It is convenient in practical applications of dynamics to express force, work, and pressure in terms of units which have reference to the force of gravity. Such units are called *gravitational units.*

In a system of gravitational units the unit of force is defined to be the weight of a definite mass, and the units of work and pressure are derived from the unit of force.

Thus English engineers use as the gravitational unit of force the **weight of one pound,** which we shall write 1 lbwt.; as the unit of work the **foot-pound,** which is the work done by 1 lbwt. acting through the distance of a foot; and as the unit of pressure the pressure of **1 lbwt. per square inch.**

We shall sometimes find it convenient to use the *weight of one gramme or of one kilogramme* as the unit of force, the *gramme-centimetre or the kilogramme-metre* as the unit of work, and the *pressure of one gramme-weight per square centimetre* as the unit of pressure.

The student of elementary dynamics knows that the weight of a given mass varies slightly from place to place on the earth's surface. The variations are, however, so small that no inconvenience results from the use of gravitational units of force, work, and pressure in practical applications of the principles of dynamics.

21. Relation between absolute and gravitational units.

It is easy to pass from absolute units to gravitational units for a given place, and conversely, when the acceleration of gravity at the place is known.

For if g denote the acceleration of gravity in foot-second units at a *given place* on the earth's surface, one pound of matter, if allowed to fall freely for one second at that place, would have a velocity of g ft./sec., and therefore a momentum of g units of momentum in the British system. But the force which generates this momentum is the force exerted by gravity on the mass of one pound, that is, is the weight of one pound.

Hence $$1 \text{ lbwt.} = g \text{ poundals},$$

and $$1 \text{ foot-pound} = g \text{ foot-poundals}.$$

Hence we get the following rules:—

(i) *Given the numerical value of a force in poundals, to find the numerical value in pounds weight—divide by the numerical value of g in foot-second units.*

(ii) *Given the numerical value of a force in pounds weight, to find the numerical value in poundals—multiply by the numerical value of g in foot-second units.*

Precisely similar rules apply to the changes in the numerical values of work, energy, and pressure due to a change from British absolute units to gravitational units, or conversely.

Also, we may convert from dynes to grammes weight by dividing by g, expressed in centimetre-second units, a number which is approximately equal to 981; and conversely, we may convert from grammes weight to dynes by multiplying by the same number. Similar rules will apply to the changes in the numerical values of work and pressure due to a change from C.G.S. absolute to gravitational units, or conversely.

Ex. 1.—A mass of 10 lbs. falls 100 feet, and is then brought to rest by penetrating one foot into sand. Find the average pressure on the sand.

The mass is acted on by its weight, 10 lbwt., through a distance of 100 feet, and therefore the work done by gravity on the mass is 1000 foot-pounds. The mass in consequence has gained this amount of kinetic energy, which is expended in doing work against the resistance of the sand. If R is the average pressure of the mass on the sand, $R \times 1$ is equal to the work done against the resistance of the sand, and this is equal to the work done by gravity on the mass.

Hence $$R = 1000 \text{ lbwt.}$$

Ex. 2.—A particle whose mass is 27 lbs. moves from rest under the action of a constant force which does on it 75 foot-pounds of work in a certain time. What is the velocity of the particle at the end of that time? [$g = 32$.]

If v denotes the velocity in ft./sec., the kinetic energy is $\frac{1}{2} \times 27 \times v^2$ in absolute units; and this, by equation (2) of Art. 18, is equal to the work done on the body. But the work done on the body is

$$75 \text{ foot-pounds}, = 75 \times 32 \text{ foot-poundals}.$$

Hence $$\tfrac{1}{2} \times 27 \times v^2 = 75 \times 32,$$

giving $$v^2 = 64 \times 25/9,$$

or $$v = 40/3, = 13\tfrac{1}{3} \text{ ft./sec.}$$

Ex. 3.—A 6-lb. cannon-ball is discharged with a velocity of 1600 feet per second. How many foot-pounds of work have been done upon the ball by

the exploding powder, and what has been the average force of propulsion if the barrel is $3\frac{1}{2}$ feet long?

What horse-power is represented by the discharge of 20 such balls per minute? [$g=32$.]

Kinetic energy of the ball is

$$\tfrac{1}{2} \cdot 6 \cdot 1600^2 \text{ foot-poundals},$$

or $$\tfrac{1}{2} \cdot 6 \cdot 1600^2/32, = 240000 \text{ foot-pounds}.$$

This, therefore, is the work done by the powder on the ball.

If R represent the average force of propulsion,

$$R \times 3\tfrac{1}{2} = 240000,$$

giving $$R = \tfrac{2}{7} \times 240000, = 68571\tfrac{3}{7} \text{ lbwt}.$$

If 20 such balls are discharged per minute, the work done per minute is 20×240000 foot-pounds, which represents a horse-power of

$$20 \times 240000/33000, = 145\tfrac{5}{11}.$$

EXAMPLES III.

[Take $g = 32$ ft./sec.^{2}]

(*The Answers are given on page* 332.)

A.

1. A force of 100 lbwt. acts on a certain mass for 3 seconds. Find the momentum of the mass at the end of that time in feet per second and pounds.

If this momentum were communicated to a free stationary mass of 125 lbs., with what velocity in feet per second would that mass move?

2. A mass of 10 lbs. moves from rest over a distance of 125 feet under the action of a force of 130 lbwt. Find the velocity in feet per second acquired by the mass.

3. A mass of 300 lbs. is moving with a velocity of 3000 yards per minute at a certain instant. From that instant it moves against a constant resistance equal to 3 lbwt. for 5 seconds. What is the kinetic energy in foot-poundals at the end of that time?

4. The momenta of two moving bodies are in the ratio of 2 to 5, and their kinetic energies are in the ratio of 4 to 15. Determine (i) the ratio of their masses, (ii) the ratio of their velocities.

5. A train of 100 tons is moving at the rate of 20 miles per hour. Determine in tons weight the constant horizontal force which would bring the train to rest (i) in 100 seconds, (ii) in 100 yards.

6. A bullet whose mass is $2\frac{1}{4}$ oz. leaves the muzzle of a gun with the velocity of 1550 feet per second. If the length of the gun barrel is $2\frac{1}{2}$ feet, find in pounds weight the average force of propulsion upon the bullet.

7. A mass of 1000 lbs., moving with a velocity of 500 feet per second, meets with an obstacle. If the obstacle yield one inch before the mass is

brought to rest, find in pounds weight the average force exerted by the mass on the obstacle.

8. A 10-ton hammer falls through a height of 6 feet, and makes an impression on the mass of iron to the extent of 1 inch. Find the average pressure on the mass of iron in pounds weight which has been exerted during the blow.

9. A body weighs 10 lbs., and moves at the rate of 1250 feet per second. Find the distance through which it could overcome a resistance of one million pounds weight.

10. It is said that a horse can do 13,200,000 foot-pounds of work in a day of 8 hours, walking at the rate of $2\frac{1}{2}$ miles per hour. What pull in pounds weight could such a horse exert continuously during the working day? How many such horses would be required to do work at the rate of 10 horse-power?

11. Prove that a train going 30 miles an hour will be brought to rest in about 84 yards by continuous brakes, supposing them to press on the wheels with a force equal to 3/4 of the weight of the train, the coefficient of friction being ·16.

12. A train of 200 tons, starting from rest, acquires a velocity of 40 miles an hour in 3 minutes on a horizontal railway. Express in tons weight the excess of the moving above the retarding forces, each being assumed to be uniform.

13. A railway train, exclusive of the engine, weighs 150 tons, and in starting along a level railway from rest, it attains a speed of 30 miles an hour in 5 minutes. What has been the mean pull between the engine and the train, the resistances being taken at 10 lbs. per ton?

14. Find the horse-power of an engine which is taking a train of 250 tons down an incline of 1 in 200 at 60 miles an hour, supposing the resistance on the level at that speed to be 35 lbs. per ton.

15. Determine the pull of the engine in the preceding question, and find how far the train will go up an incline of 1 in 200 before the velocity falls from 60 to 15 miles per hour, supposing the pull of the engine and the resistances to remain unaltered.

16. What must be the effective horse-power of a locomotive engine which moves at the steady speed of 40 miles per hour on a level railway, the resistances being 15 lbs. per ton, and the mass of the engine and train being 100 tons?

If the rails were laid at a gradient of 1 in 100, what additional horse-power would be required?

B.

17. The head of a steam-hammer weighs 50 cwt., and there is a drop of 5 feet. What will be the average force of compression during a blow from this hammer on the supposition that the duration of the blow is 1/80 of a second?

18. A particle, falling under gravity, is moving at a certain instant with the velocity of 116 ft./sec. How long will it take to describe the next 100 feet, the resistance of the air being neglected?

If, owing to the resistance, it takes ·9 of a second, find the ratio of the resistance (assumed to be constant) to the weight of the particle.

19. A body, whose mass is 10 lbs., slides down an inclined plane whose height is 100 feet, and reaches the foot of the plane with a velocity of 60 feet per second. How many foot-pounds of work have been expended during the motion on friction and other resistances?

20. A particle whose mass is m lbs. moves from rest under the action of a force which does on it W foot-pounds of work. What is its velocity at the end of that time?

21. Compare the momenta and also the kinetic energies of two unequal masses, m and m' respectively, when acted on by equal forces—(i) for equal times (ii) over equal distances.

22. Prove that the resistance of the wood is 204 pounds weight to a nail weighing 1 oz., supposing that a hammer weighing 1 lb. and striking the nail with a velocity of 34 ft./sec. drives the nail one inch into a fixed block of wood.

If the block is free to move and weighs 68 lbs., prove that the hammer will drive the nail only 64/65 of an inch.

23. A column of water, a quarter of a square foot in section, descends on a horizontal inelastic area at the rate of 3000 gallons a minute. If the water runs freely off the area, find the pressure on the area in pounds per square foot.

[A gallon of water weighs 10 lbs.]

24. If the unit of work be 2520 foot-pounds, the unit of force the weight of a mass of 784 lbs., and the unit of time 3 seconds; find the units of mass and distance.

25. Two particles are connected by an inextensible thread which passes over a smooth point. One of the particles has a mass of 5 lbs., and is at rest on a table; the other particle has a mass of 3 lbs. and descends. If the descending particle falls through 10 ft. before the string is drawn straight, how much of the kinetic energy disappears when the thread is drawn straight?

CHAPTER III.—DENSITY AND SPECIFIC GRAVITY.

22. Density.

Def. *The density, supposed to be uniform, of a substance is the mass of unit volume of the substance.*

It follows from this definition that the number measuring

the density of a substance depends on the units of mass and volume.

If the unit of mass is one pound, and the unit of volume one cubic foot, the density is measured by the number of pounds in a cubic foot of the substance.

If the C.G.S. system of units is being used, the unit of mass is one gramme, the unit of volume is one cubic centimetre, and the number measuring the density of a substance is the number of grammes in a cubic centimetre of the substance.

If the density of the substance is not uniform, the density at a given point of the substance would be found by taking a very small volume inclosing the point, and dividing the mass of this small volume by the fraction which this small volume is of unit volume. The number which is thus obtained measures the density, and evidently represents what would be the mass of unit volume of the substance if the density were uniform and equal to that at the point under consideration.

23. Formula connecting Mass, Volume, and Density.

If ρ represent the density of a substance, supposed to be uniform, and M the mass of V units of volume of the substance, then

$$M = V\rho.$$

For, by definition, ρ is the mass of one unit of volume, and therefore the mass of V units of volume is $V\rho$.

This equation may be written in two other forms:—

$$\rho = M/V \text{ and } V = M/\rho.$$

From the first of these forms we see that *the density of a given mass of a substance varies inversely as the volume of the mass.*

A change in the temperature of a body produces a change in its volume, and therefore the density of a body depends on its temperature.

The changes of density in solids and liquids due to changes of temperature are in general small, and will be left out of account in this book. Also all solids and liquids will be considered to be of uniform density unless the contrary is expressly stated.

In the case of gases, however, changes of temperature under

certain circumstances produce changes of density which cannot be neglected. The relation between temperature and density in a gas forms one of the important laws of gases.

24. Density of water.

It is found by experiment that the density of water is greatest at the temperature 4° on the Centigrade scale, or, what is the same thing, at the temperature 39°·2 on the Fahrenheit scale. This temperature is often referred to as the *maximum density point of water.*

It is also found by experiment that—

I. *The mass of a cubic foot of pure water at its maximum density point is nearly* 1000 *oz. or* 62·5 *lbs.;*

II. *The mass of a cubic centimetre of pure water at its maximum density is* 1 *gramme.*

Hence in British units the density of water at the point of maximum density is 62·5, and in C.G.S. units is 1. In this book we shall always suppose that water is at this temperature.

Since the density of water in the C.G.S. system is unity, it follows that—

III. *The volume of a mass of* m *grammes of water is* m *cubic centimetres.*

Ex. 1.—The mass of a cubic foot of a substance A is 62·5 lbs., and the mass of 36 cubic inches of another substance B is 75 oz. Compare the density of A with the density of B.

The mass of a cubic foot of A is 62·5 lbs.

The mass of a cubic foot of B is

$$75 \times 1728/36 \text{ oz.}, = 75 \times 1728/(36 \times 16) \text{ lbs.}$$

Hence density of A : density of B

$$= 62{\cdot}5 : \frac{75 \times 1728}{36 \times 16}, = 5 : 18.$$

Ex. 2.—A cylinder, whose base is a circle 1 foot in diameter, and whose height is 3 feet, weighs 10 lbs. Calculate its density in pounds per cubic foot.

To find the volume, V, in cubic feet, we use the formula

$$V = \pi r^2 h, \qquad \text{(Art. 14)}$$

putting $r = 1/2$, $h = 3$. Taking $\pi = 22/7$, we get

$$V = \tfrac{22}{7} \times \tfrac{1}{4} \times 3, = \tfrac{33}{14} \text{ cubic feet.}$$

The mass, $M, = 10$ lbs.

Hence the density, ρ, is

$$= M/V, = 10/\tfrac{33}{14}, = \tfrac{140}{33},$$

$$= 4\tfrac{8}{33} \text{ pounds per cubic foot.}$$

25. Density relative to water.

Def. *The ratio of the density of a substance to the density of water is called the density of the substance relative to water.*

From this definition it immediately follows that the density of a substance relative to water is equal to the ratio of the mass of a given volume of the substance to the mass of the same volume of water. For if V be a given volume of a substance whose density is ρ, and if σ be the density of water, then the mass of volume V of the substance is $V\rho$, and the mass of the same volume of water is $V\sigma$. Hence the ratio of the mass of volume V of the substance to the mass of the same volume of water is

$$V\rho : V\sigma, \; = \rho : \sigma, = \text{density of substance relative to water.}$$

Also, since

$$\frac{\text{absolute density of substance}}{\text{absolute density of water}} = \text{density of substance relative to water,}$$

it follows that

absolute density of substance

$= \text{absolute density of water} \times \text{density of substance relative to water.}$

Hence, since the density of water in the C.G.S. system is unity, it follows that the density of a substance in the C.G.S. system is equal to its density relative to water. Thus if s denotes the density of a substance relative to water, then s is also the density of the substance in grammes per cubic centimetre.

26. Specific gravity.

The **specific gravity** of a substance is the ratio of the density of the substance to the density of some standard substance.

The standard substance is usually taken to be water, so that the term specific gravity is usually used in the sense of density relative to water; and it is in this sense that the term will be used in this book.

A table of specific gravities, or densities relative to water, is given on page vi. This table is also a table of absolute densities in grammes per cubic centimetre.

We must warn the student that in defining the specific gravity of a body

to be its density relative to water we have departed from the definition formerly given in books on hydrostatics. It was formerly usual to define specific gravity as the ratio of the *weight* of a given volume of a substance to the *weight* of the same volume of water; and it was then stated that, since the weights of bodies are proportional to their masses, the specific gravity, so defined, is equal to the ratio of the mass of a given volume of the substance to the mass of the same volume of water. Thus the older definition leads to the same result as the definition—density relative to water. In fact, in the equation $M = V\rho$ we may consider M to be the *weight* of a volume V of a substance, the *weight* of unit volume of which is ρ. In a similar way we may speak of the *weight* of a body as found by the balance, although what is actually determined by weighing a body in a balance against standard masses is the *mass* and not the *weight* of the body.

Ex. 1.—The length of a copper wire is 1200 feet, and its sectional area is 1/10 of a square inch. Taking the specific gravity of copper to be 8·7 find the mass of the wire in pounds.

The volume of the wire in cubic feet

$$= 1200 \times \tfrac{1}{10} \times \tfrac{1}{144},$$
$$= 5/6.$$

The density of copper, in pounds per cubic foot,

$$= 8{\cdot}7 \times \text{density of water}, = 8{\cdot}7 \times 62{\cdot}5.$$

Hence by the formula $M = V\rho$, we find that the mass of the wire

$$= \tfrac{5}{6} \times 8{\cdot}7 \times 62{\cdot}5, = 453{\cdot}125 \text{ lbs.}$$

Ex. 2.—What is the capacity in cubic centimetres of a vessel which holds 81576 grammes of mercury at a certain temperature, the density of mercury at that temperature being 13·596?

Since the specific gravity of mercury is 13·596, the mass of a cubic centimetre of mercury is 13·596; and, since the vessel contains 81576 grammes of mercury, the capacity of the vessel

$$= 81576/13{\cdot}596 = 6000 \text{ cubic centimetres.}$$

27. Specific gravity of a mixture of two or more fluids.

A number of fluids of given specific gravities are mixed together to form a fluid of uniform density; it is required to find the specific gravity of the mixture.

The mixture may be formed by taking given volumes of the fluids, or by taking given masses of the fluids. Those two cases we consider separately.

I. Mixture by volumes.

Let a mixture be made of volumes $v_1, v_2, v_3 \ldots$ of fluids whose specific gravities are $s_1, s_2, s_3 \ldots$ respectively. It is required to find the specific gravity of the mixture.

Let w denote the density of water, that is, the mass of unit volume of water; and let V represent the volume of the mixture. If there is no contraction, then V is equal to the sum of the volumes v_1, v_2, v_3...; but if there is contraction, as when water and alcohol are mixed together, V must be found by experiment.

The *mass* of the mixture is equal to the sum of the masses of the fluids mixed.

The masses of volumes of water equal to the volumes of the fluids are

$$wv_1,\ wv_2,\ wv_3, \dots \text{ respectively,}$$

and therefore the masses of the fluids are

$$ws_1v_1,\ ws_2v_2,\ ws_3v_3, \dots \text{ respectively.}$$

Hence the sum of the masses of the fluids is

$$w(s_1v_1 + s_2v_2 + s_3v_3 + \dots).$$

But if s denote the specific gravity of the mixture, the mass of the mixture is wVs. Hence

$$wVs = w(s_1v_1 + s_2v_2 + s_3v_3 + \dots),$$

from which

$$s = (s_1v_1 + s_2v_2 + s_3v_3 + \dots)/V.$$

This formula may be written in a shorter form by using the Σ notation. Write Σsv for $s_1v_1 + s_2v_2 + s_3v_3 + \dots$, and the formula becomes

$$s = \Sigma sv/V.$$

If there is no change of volume produced by the mixture,

$$V = v_1 + v_2 + v_3 + \dots, = \Sigma v,$$

and the formula for s becomes in this case

$$s = \Sigma sv/\Sigma v.$$

II. **Mixture by masses.**

Let a mixture be made of fluids, whose masses are w_1, w_2, w_3,... and specific gravities s_1, s_2, s_3,... respectively; and let s, w, V denote the same quantities as before.

The mass of the mixture is

$$= w_1 + w_2 + w_3 + \dots\dots\dots\dots,\ = \Sigma w.$$

But the mass of volume V of specific gravity s is $w\,s\,V$; and these two masses are equal. Hence

$$w s V = \Sigma w,$$

from which $$s = \Sigma w / w V.$$

If there is no change of volume,

$$\begin{aligned} V &= \text{sum of volumes of fluids,} \\ &= w_1/w\,s_1 + w_2/w\,s_2 + w_3/w\,s_3 + \ldots\ldots \\ &= \frac{1}{w}\,\Sigma\frac{w}{s}. \end{aligned}$$

Hence $\quad w\,V = \Sigma\,(w/s),$

and the formula for s becomes

$$s = \Sigma w / \Sigma (w/s).$$

Ex. 1.—Three parts by volume of a liquid of specific gravity ·8 are mixed with seven parts of water, and the mixture shrinks in the ratio of 25 to 24. Find the specific gravity of the mixture.

Here $v_1 = 3$, $v_2 = 7$, $s_1 = {}$·8, $s_2 = 1$, and the volume $V = 24 \times 10/25$. Hence

$$\frac{24 \times 10}{25} \times s = 3 \times \tfrac{4}{5} + 7 \times 1;$$

from which $$s = 47/48.$$

Ex. 2.—The specific gravity of gold being 19·26, and that of silver 10·47, calculate the amount of gold in an alloy of those metals whose specific gravity is 16 and mass 650 grains.

Let w_1 and w_2 be the masses of gold and silver respectively in the alloy. Then we have the following equations:—

$$w_1 + w_2 = 650,$$

$$\frac{w_1}{19{\cdot}26} + \frac{w_2}{10{\cdot}47} = \frac{650}{16}.$$

Solving these simultaneous equations for the two unknowns w_1 and w_2, we get

w_1, the weight of gold = 492·25 grains,

w_2, the weight of silver = 157·75 grains.

EXAMPLES IV.

[It is assumed that—

1 cubic foot of water weighs 1000 oz.;

1 cubic centimetre of water weighs 1 gramme;

$\pi = 22/7$.]

(*The Answers are given on page* 332.)

A.

1. In a certain state of the atmosphere 100 cubic inches of air weigh 31 grains. If 30 cubic inches of mercury weigh 14·88 pounds, find the ratio of the mass of a cubic inch of mercury to the mass of a cubic inch of air.

2. Under certain conditions of pressure and temperature 100 cubic inches of oxygen weigh 35 grains. If one cubic inch of mercury weighs ·49 of a pound, how many cubic inches of oxygen would contain as much matter as a cubic inch of mercury?

3. One body, A, has a volume of 1·35 cubic feet and a specific gravity of 4·4. Another body, B, has a volume of 10·8 cubic inches and a specific gravity of 19·8. What ratio does the quantity of matter in A bear to that in B?

4. Of two bodies one has a volume of 5 cubic inches and the other of 1/5 of a cubic foot. The mass of the first body is 15 oz., and of the second 12·8 lbs. What is the ratio of the mass of the first to the mass of the second? What is the ratio of the density of the first to the density of the second?

5. A rectangular block of ice has a length of 3 feet, a breadth of 1 foot 8 inches, and a height of 2 feet 6 inches. Find its mass in lbs., the specific gravity of ice being ·92.

6. A cylinder, the base of which is 6 inches in radius, and which is 14 feet high, is filled with mercury whose specific gravity is 13·6. Find the mass of the mercury.

7. A rod of uniform section, 27 inches long, weighs 3 lbs., its specific gravity being 8·8. What is the area of its cross section?

8. A rod of uniform section, 18 inches long, weighs 3 oz., the specific gravity being 8·8. What fraction of a square inch is its section?

9. Find the cross section of a cylinder, 120 centimetres long and specific gravity 5, which weighs 250 grammes.

10. A piece of copper wire 4 metres long weighs 1720 grammes. Taking the specific gravity of copper to be 8·8, find the diameter of the wire.

11. If two pints of water and three pints of alcohol are mixed, find the specific gravity of the mixture assuming that there is no contraction. Specific gravity of alcohol is ·8.

12. If 15 oz. of a liquid, whose specific gravity is 1·07, are mixed with 20 oz. of another liquid whose specific gravity is ·86, what is the specific gravity of the mixture, there being neither contraction nor expansion?

13. A certain mass of liquid, whose specific gravity is ·5, is mixed, without suffering contraction, with four times that mass of a second liquid whose specific gravity is 1·25. Find the specific gravity of the mixture.

14. Two volumes of alcohol, specific gravity ·793, are mixed with one volume of water, and the total volume on mixture and cooling contracts 1·5 per cent. Find the specific gravity of the mixture.

15. The specific gravity of gold is 19·3, and that of silver 10·4. What is the composition by weights of an alloy of gold and silver whose specific gravity is 17·6, no change of volume being supposed to accompany the admixture of the metals?

16. Two equal spheres, A and B, contain air, the density of the air in A

being double the density of the air in B. If half the quantity of air in B is transferred to A, what is then the ratio of the densities of the air in the two spheres?

B.

17. A mixture formed of equal weights of two fluids is divided into two parts, and to each part is added its own weight of one of the fluids. The specific gravities of the mixtures so formed are in the ratio of 3 to 5. Find the ratio of the specific gravities of the two fluids.

18. A mixture formed of equal weights of three fluids is divided into three parts, and to each part is added its own weight of one of the fluids. The specific gravities of the mixtures so formed are as $5 : 6 : 7$. Find the ratios of the specific gravities of the three fluids.

19. A mixture formed of equal weights of three fluids is divided into three parts, and to each part is added its own weight of one of the fluids. The specific gravities of the fluids so formed are as $3 : 4 : 5$. Find the ratios of the specific gravities of the three fluids.

20. In what proportions (i) by volumes, (ii) by masses, must two liquids of given specific gravities s and s' respectively be mixed in order that the specific gravity of the mixture may be the arithmetic mean of s and s'?

21. In what proportions (i) by volumes, (ii) by masses, must the two liquids in the preceding question be mixed in order that the specific gravity of the mixture may be the harmonic mean between s and s'?

22. The specific gravity of a mixture of a gallon of A with 32 lbs. of B is 24/25, of a mixture of a gallon of A with 64 lbs. of B is 8/9, and of a mixture of a gallon of A with 96 lbs. of B is 56/65. Find from these data the weight of a gallon of A and the specific gravities of A and B.

23. The density of a substance A relative to another substance B is k, and the density of B relative to a third substance C is k'. What is the density of A relative to C?

CHAPTER IV.—PRINCIPLES OF STATICS.

28. The fundamental proposition in Statics is the Parallelogram of Forces. Assuming that the student is familiar with this proposition, with the method of finding the resultant of two or more parallel forces, and with the principle of moments, we proceed to consider very briefly the general theory of forces in one plane and of the centre of gravity. The results which we shall arrive at are required in the study of fluid pressure.

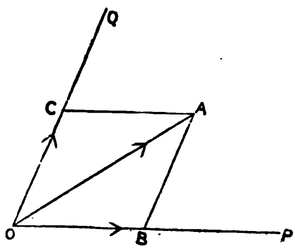
Concurrent forces, Arts. 29 to 32.

29. **Resolution of a force into two components in two given directions.**

Def.—*The components of a given force in two given directions are the forces acting in the two given directions which have the given force for their resultant.*

Let the straight line O A represent a force in magnitude and direction; and let O P and O Q be two given directions.

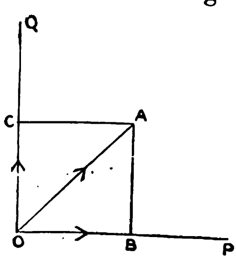
From A draw A B parallel to Q O, and A C parallel to P O, meeting O P and O Q in B and C respectively. Then the figure O B A C is a parallelogram by construction.



By the parallelogram of forces, the force O A is the resultant of the two forces O B and O C. Hence O B and O C are the components of the given force in the directions O P and O Q.

30. **Resolution of a force into two rectangular components in two given directions.**

When the two given directions O P and O Q are at right angles to each other, the two forces O B and O C acting along O P and O Q respectively which have the given force O A for their resultant, are called the **rectangular components** of the given force in the directions O P and O Q.



Let F denote the magnitude of the force O A, and θ the angle A O P which O A makes with the direction O P. Then, since the triangle A O B is right-angled at B, we have, by trigonometry,

$$OB = OA \cos AOB, = OA \cos \theta,$$

and

$$OC = BA, \qquad = OA \sin \theta.$$

Therefore the rectangular components of the given force F in the given directions OP and OQ are respectively

$$F \cos \theta \text{ and } F \sin \theta.$$

The expression—*component of a given force in a given direction*—is often used. By this is meant the force acting in the given direction, which, compounded with another force acting in the direction at right angles to the given direction, gives a resultant equal to the given force. Thus the component of F in the direction OP is

$$F \cos \theta, = \text{force} \times \text{cosine of angle between the direction of the force and the direction OP}.$$

Thus we have the following result:—

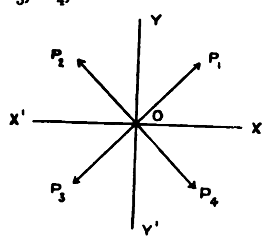
The component of a force in a given direction is found by multiplying the force by the cosine of the angle between the direction of the force and the given direction.

31. Resultant of any number of concurrent forces.

Let any number of forces $P_1, P_2, P_3, P_4, \ldots$ act at a point O. It is required to find their resultant.

Take two lines XOX′ and YOY′ at right angles through the point O.

Resolve each of the forces $P_1, P_2, P_3, P_4, \ldots$ into two components, acting one in the line XOX′ and the other in the line YOY′. And in finding these components let us make use of the signs + and − to distinguish between the two directions in which a force may act in the same straight line. Take OX as the positive direction in the line XOX′,



and OY as the positive direction in the line YOY′. Then the component of a force P in the line X′OX will be positive if it acts in the direction from O to X, and will be negative if it acts in the direction from O to X′; and the component of the force P in the line YOY′ will be positive if it acts in the direction from O to Y, and will be negative if it acts in the direction from O to Y′.

Let X_1, Y_1; X_2, Y_2; X_3, Y_3; X_4, Y_4;... denote respectively the components of the forces of the system in the directions OX and OY. In the above figure X_1 and X_4 would be positive, but X_2 and X_3 would be negative; also, Y_1 and Y_2 would be positive, but Y_3 and Y_4 would be negative.

Thus the given system of forces $P_1, P_2, P_3, P_4, \ldots$ may be replaced by two systems of forces:—

$X_1, X_2, X_3, X_4 \ldots$ acting in the line XOX′,

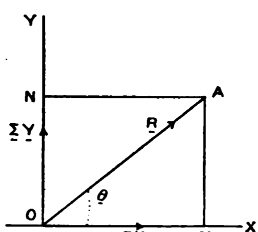
and

$Y_1, Y_2, Y_3, Y_4 \ldots$ acting in the line YOY′.

We know from the elements of Statics that the resultant of a system of forces in the same straight line is found by taking the algebraical sum of the forces. Let ΣX and ΣY denote the algebraical sums of the forces in the lines XOX′ and YOY′ respectively, then the given system of forces may be replaced by ΣX and ΣY acting along OX and OY respectively, where

$$\Sigma X = X_1 + X_2 + X_3 + X_4 + \ldots,$$
$$\Sigma Y = Y_1 + Y_2 + Y_3 + Y_4 + \ldots$$

Lastly, find the resultant of the forces ΣX and ΣY acting in the lines OX and OY.



Let OM and ON represent ΣX and ΣY, and let the parallelogram OMAN be completed. Then OA represents the resultant of ΣX and ΣY, that is, OA represents the resultant of the given system of forces $P_1, P_2, P_3, P_4, \ldots$

Denoting OA by R, and the angle AOM by θ, we get

$$R^2 = \text{OA}^2, = \text{OM}^2 + \text{MA}^2, = \text{OM}^2 + \text{ON}^2,$$
$$= (\Sigma X)^2 + (\Sigma Y)^2;$$

giving

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2} \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, $\tan\theta = \text{AM}/\text{OM}, = \text{ON}/\text{OM},$

$$= \Sigma Y/\Sigma X \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Equations (1) and (2) determine completely the resultant of the given system of forces.

32. Conditions of equilibrium of concurrent forces. In order that the system of forces $P_1, P_2, P_3, P_4, \dots$ may be in equilibrium, it is sufficient and necessary that the value of R found in the preceding article shall be zero.

Now since R represents the diagonal of a rectangle constructed on ΣX and ΣY as adjacent sides, it is evident that R will not be zero unless both ΣX and ΣY are zero.

Hence we infer that the conditions of equilibrium are

$$\Sigma X = 0 \text{ and } \Sigma Y = 0.$$

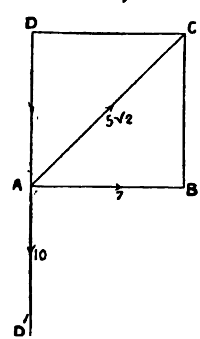
Since the two directions OX and OY were chosen arbitrarily, it follows that the sufficient and necessary **conditions for the equilibrium of a system of concurrent forces** are—

The algebraical sums of the components of the forces in any two directions at right angles must be zero.

These conditions may be applied to the solution of all statical problems in which the forces are concurrent.

If one of the forces in the problem is the force of gravity, it will be found convenient to resolve the forces in the horizontal and vertical directions. For equilibrium it is sufficient and necessary that the algebraical sums of the horizontal and vertical components shall each be zero.

Ex. 1.—ABCD are the angular points of a square taken in order. Find the magnitude of the resultant of the three forces, 7 from A to B, 10 from D to A, $5\sqrt{2}$ from A to C.



Here we have three forces acting at the point A. Producing DA to D′, find the sum of the components of the forces along AB and AD′.

The force $5\sqrt{2}$ acting along AC has a component $5\sqrt{2}\cos 45°$, or 5, along AB, and a component $5\sqrt{2}\cos 45°$, or 5, along AD, or -5 along AD′.

Hence the sum of the components along AB is

$$7 + 5, = 12;$$

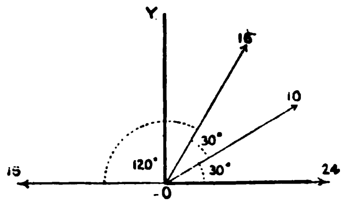
and the sum of the components along AD′ is

$$10 - 5, = 5.$$

The three forces are therefore equivalent to two forces, 12 along AB and 5 along AD′. The resultant of these two forces, which act in directions at right angles to each other, is

$$\sqrt{12^2 + 5^2}, = 13.$$

Ex. 2.—Four forces of 24, 10, 16, 16 units act upon a particle, the angle between the first and second being 30°, the angle between the second and third 30°, and the angle between the third and fourth 120°, the angles being all measured in the same direction. Find the magnitude and direction of the resultant.



Let OX be the direction of the first force and OY the direction at right angles to OX. Resolve each of the forces in these two directions.

Using the notation of Article 31, we get

$$X_1=24; \qquad Y_1=0;$$
$$X_2=10 \cos 30°, =5\sqrt{3}; \quad Y_2=10 \sin 30°, =5;$$
$$X_3=16 \cos 60°, =8; \qquad Y_3=16 \sin 60°, =8\sqrt{3};$$
$$X_4=-16; \qquad Y_4=0.$$
$$\Sigma X=X_1+X_2+X_3+X_4=24+5\sqrt{3}+8-16,$$
$$=16+5\sqrt{3};$$
$$\Sigma Y=0+5+8\sqrt{3}+0,$$
$$=5+8\sqrt{3}.$$

Hence if R denote the magnitude of the resultant, and θ the angle which its direction makes with OX,

$$R^2=(16+5\sqrt{3})^2+(5+8\sqrt{3})^2,$$
$$=548+240\sqrt{3};$$

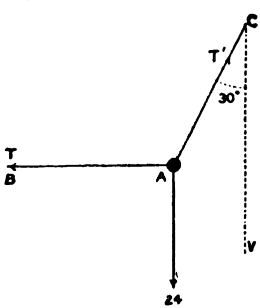
therefore

$$R=\sqrt{548+240\sqrt{3}}, =31 \text{ nearly}.$$

Also $\tan\theta=\Sigma Y/\Sigma X=(5+8\sqrt{3})/(16+5\sqrt{3})$,

$$=(103\sqrt{3}-40)/181, =\cdot 765.$$

From a table of natural tangents we find that the angle (to the nearest degree) whose tangent is ·765 is 37°.



Hence the resultant is approximately 31, and its direction makes an angle of 37° with the force 24.

Ex. 3.—A ball weighing 24 lbs. is kept at rest by two threads, one of which is horizontal and the other inclined to the vertical at an angle of 30°. Find the tensions of the two threads.

Let A be the ball, which is kept at rest by the tension, T say, of the horizontal thread A B, and the tension, T' say, of the thread A C, which is inclined to the vertical C V at an angle of 30°.

Since the ball is in equilibrium, the algebraical sums of the components of the forces in the horizontal and vertical directions must each be zero.

The algebraical sum of the horizontal components

$$T - T' \cos 60^\circ, = T - T'/2,$$

and of the vertical components is

$$24 - T' \cos 30^\circ, = 24 - T' \sqrt{3}/2.$$

Equating these expressions to zero, we get

$$T - T'/2 = 0,$$
$$24 - T' \sqrt{3}/2 = 0.$$

The second equation gives

$$T' = 48/\sqrt{3}, = 16\sqrt{3} \text{ pounds weight,}$$

and this, substituted in the first equation, gives

$$T = 8\sqrt{3} \text{ pounds weight.}$$

EXAMPLES.

1. Five forces 18, $12\sqrt{2}$, 50, $3\sqrt{2}$, and $12\sqrt{2}$ act respectively east, north-east, north, north-west, and west. Find the effective eastern component.

Ans. $27 - 12\sqrt{2}$.

2. Find the magnitude of the resultant of each of the following systems of concurrent forces:—

(i) Forces 5, 10, 15, 20, acting respectively east, north, west, and south.

(ii) Forces 1, 2, 3, 4, 5, 6, acting from the centre of a regular hexagon to the angular points of the hexagon taken in order.

(iii) Forces 3, 4, 5, parallel to the sides of an equilateral triangle taken in order.

(iv) Forces 3, 5, 7, acting at the centre of a circle towards points on the circumference of the circle which divide it into three equal parts.

(v) Forces 30, 40, 40, the angle between the directions of the first two forces being 45°, and that between the second and third forces (measured in the same direction) being 30°.

(vi) Forces P, P, $2P$, parallel to the sides of an equilateral triangle taken in order.

(vii) Forces of $3\sqrt{2}P$, $4P$, $5P$, $6P$, $2P$, the directions of the last four forces making angles of 45°, 135°, 225°, 315°, respectively with the first force, all the angles being measured in the same direction.

Ans. (i) $10\sqrt{2}$. (ii) 6. (iii) $\sqrt{3}$. (iv) $2\sqrt{3}$. (v) 96·6. (vi) P. (vii) P.

3. Draw an isosceles triangle ABC with the angle A equal to 120°, and draw AD perpendicular to BC. A force P acts from A to B, a force $2P$ from A to C, and a force $2P$ from D to A. Find the magnitude of the resultant.

Ans. P.

4. A mass of 10 lbs. is supported by two threads, one horizontal and the other inclined to the vertical. Find, to the first place of decimals, the tensions of the two threads in the cases in which the inclination to the vertical of the thread which is not horizontal is (i) 30°, (ii) 45°, (iii) 60°.

Ans. (i) 11·5 and 5·8 lbwt. respectively. (ii) 14·1 and 10 lbwt. respectively. (iii) 20 and 17·3 lbwt. respectively.

5. A mass of 24 lbs. is supported by two threads, which are both inclined to the vertical. Find the tensions of the two threads when the inclinations of the two threads to the vertical are—(i) 30° and 60° respectively, (ii) 45° and 45° respectively.

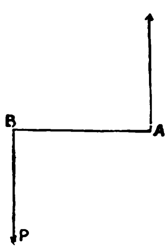
Ans. (i) 12 and 20·8 lbwt. respectively. (ii) 17 and 17 lbwt. respectively.

Forces not concurrent, Arts. 33 to 37.

33. Couples.

Def.—*A couple in dynamics is a system of two parallel forces which are equal in magnitude and act in opposite directions.*

If P represent either force of a couple, and if A B be the perpendicular distance between the lines of action of the two forces, then AB is called the **arm** of the couple, and the product of the force P and the arm A B is called the **moment** of the couple.

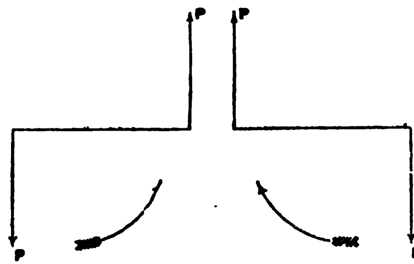


If we attempt to find the resultant of the two forces forming a couple by the rule for compounding two parallel forces, we find that the method fails. It follows that a single force cannot be found which is equivalent to the two forces forming a couple. Now a single force, acting on a body, tends to cause a motion of translation in the straight line in which the force acts. The kind of motion which a couple causes, or tends to cause, must therefore be different from a motion of translation; it can therefore only be a motion of rotation.

An example of a couple is seen in the effect of the earth's magnetism on a compass-needle. The needle rests in the position in which it lies due magnetic north and south. If displaced from this position, the earth's magnetism attracts one end of the needle, and repels the other with equal parallel forces, and those two forces form a couple tending to make the needle rotate back to its former position.

By the simple experiment of floating a compass-needle on a piece of cork in water, it may be shown that those two forces have no translational effect on the needle.

A couple may tend to produce rotation in the body on which it acts either in the direction of the hands of a watch or in the opposite direction. In the accompanying diagram the couple on the left tends to produce rotation in the direction contrary to that of the hands of a watch, while the couple on the right tends to produce rotation in the opposite direction.



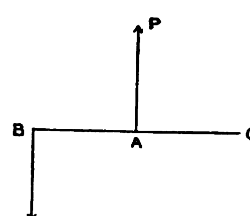
The **sense** of a couple is the direction in which it tends to produce rotation. It is convenient to call one direction of rotation the positive direction, and the opposite direction the negative direction; and, further, to represent the moments of couples whose senses are negative by negative numbers. By adopting this convention the statement of certain propositions regarding couples can be simplified.

34. Elementary proposition regarding couples.

PROP. I.—*The algebraical sum of the moments of the forces forming a couple about any point in the plane of the couple is constant, and equal to the moment of the couple.*

Let P, P be the two forces forming a couple, and let O be any point in the plane of the couple.

Draw OAB perpendicular to the lines of action of the forces. Then AB is the arm of the couple, and $P \times AB$ is the moment of the couple.



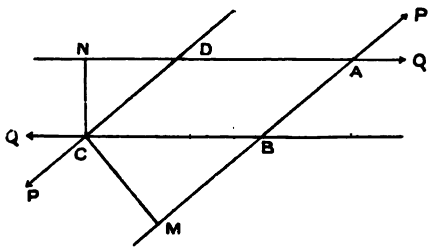
The algebraical sum of the moments of the forces P, P about O, the direction contrary to that of the hands of a watch being taken as the positive direction, is

$$=P \times OB - P \times OA, = P\,(OB - OA),$$
$$=P \times AB, = \text{moment of the couple.}$$

PROP. II.—*Two couples of opposite senses acting in the same plane will balance if their moments are equal.*

Let P, P, acting in the lines BA and DC, form a couple, and let Q, Q acting in the lines DA and BC form a couple opposite in sense and of equal moment. Then the two couples will balance.

We have to prove that the four forces P, P, Q, Q are in equilibrium. Let their lines of action meet in the points A, B, C, D; then ABCD is a parallelogram. Let CM and CN be drawn perpendicular to AB and CD, the lines of action of P, P and Q, Q respectively.



The resultant of the two forces P and Q acting at A is a force, R say, and the resultant of the forces P and Q acting at C will be a force equal to R, since the angle between the forces P and Q acting at A is equal to the angle between the forces P and Q acting at C. (Euclid I. 34.)

Since the moments of the couples are by supposition equal,

$$P \times CM = Q \times CN,$$

therefore $$P : Q = CN : CM,$$
$$= CD : CB \quad \text{(by similar triangles CDN and CBM).}$$

Hence CD and CB may be taken to represent respectively the forces P and Q which act at A, for the directions of CD and CB are respectively those of P and Q, and their magnitudes are proportional to P and Q. It follows, by the parallelogram of forces, that CA is the direction of the resultant R. Hence the resultant of the forces P and Q acting at A is a force R acting from C to A in the line CA.

Similarly it may be shown that the resultant, R, of the forces P and Q acting at C acts from A to C in the line CA. Hence the four forces P, P, Q, Q are equivalent to two equal forces R, R, acting in opposite directions in the line CA, and are therefore in equilibrium.

Hence the two couples balance.

PROP. III.—*Two couples in the same plane of the same sense and of equal moments are equivalent.*

Let there be a couple in a plane whose moment is M, and let AB be any line in the plane. Let a force P be found such that $P \times AB = M$.

Apply at A two forces, P and P, acting in opposite directions in the line perpendicular to AB, and apply similarly at B forces, P and P, acting perpendicular to AB and in opposite directions. This does not alter the effect of the given couple, as the four forces thus introduced will form a balancing system of forces. These four forces are equivalent to two couples of opposite senses, whose moments are each equal to M; and one of the couples so formed must therefore (Prop. II.) balance the given couple. Removing the two balancing couples, we get as the equivalent of the given couple a couple whose moment is $P \times AB = M$, and whose sense is the same as the sense of the given couple.

From this proposition it follows that a couple may be moved in its own plane from one position to any other without altering its effect. Also we may change the arm of the couple if at the same time we change the force, so that the moment remains unaltered.

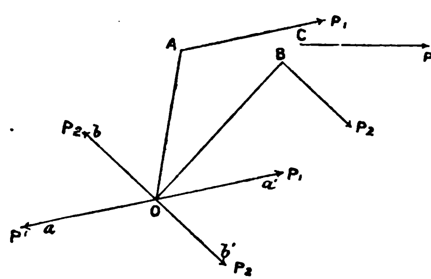
PROP. IV.—*The resultant of any number of couples acting in one plane is a couple whose moment is equal to the algebraical sum of the moments of the couples.*

The couples whose moments are M_1, M_2, M_3,... may, by Prop. III., be replaced by couples of equal moments respectively acting at the same arm. Let this arm be taken of unit length; then the forces of the couples replacing the given couples will be M_1 and M_1, M_2 and M_2, M_3 and M_3,... respectively. These forces will form a couple whose arm is unity, whose forces are each equal to $M_1 + M_2 + M_3 + \ldots$, and whose moment is therefore $M_1 + M_2 + M_3 + \ldots$

Hence the given couples are equivalent to a single couple whose moment is the algebraical sum of the moments of the given couples.

35. Resultant of a system of forces acting on a body in one plane.

Let P_1, P_2, P_3,... be a system of given forces acting in the same plane at given points A, B, C,... of a body respectively. It is required to find their resultant.



Take any point O in the plane of the forces, and draw through O lines $a\,a'$, $b\,b'$... parallel respectively to the lines of action of the given forces P_1, P_2,... Let p_1, p_2, p_3,... be the lengths of the perpendiculars from O upon the lines of action of the forces P_1 at A, P_2 at B, P_3 at C... respectively.

Superimpose upon the body a pair of forces each equal to P_1, acting in opposite directions in the line $a\,a'$. These forces form a balancing system, and therefore do not alter the effect of the given system of forces. Superimpose, in the same way, a pair of balancing forces each equal to P_2, acting in the line $b\,b'$; and do the same in the case of each of the lines through O.

Of the three forces P_1 at A, P_1 acting along Oa and P_1 acting along Oa', the first two form a couple whose moment is $P_1 p_1$. Hence the force P_1 at A is equivalent to, and may be replaced by,

a force P_1 along Oa' and a couple whose moment is $P_1 p_1$.

Similarly the given force P_2 at B may be replaced by a force

P_2 along Ob' and a couple whose moment is $P_2 p_2$.

In the same way each of the given forces P_3, P_4... may be replaced by an equal and parallel force, and a couple whose moment is force × perpendicular from O upon the given force.

The forces P_1, P_2,... acting in the lines Oa', Ob'... can be compounded into a force, $=R$ say, and the couples may be compounded into a couple whose moment is equal to the algebraical sum of the moments of the couples, that is, equal to

$$P_1 p_1 + P_2 p_2 + \ldots \qquad , = M \text{ say.}$$

The force R is the force which would be the resultant of the given system of forces if the point of application of each force were transferred to the point O, the magnitude and direction of each force being unaltered.

The moment M of the resultant couple is equal to $P_1 p_1 + P_2 p_2 + \ldots$, that is, is equal to the algebraical sum of the moments of the given forces about O.

The moment M of the resultant couple will, in general, depend on the position of the point O, but the magnitude and direction of the resultant force R will be the same for all positions of O.

36. Conditions of equilibrium of a system of forces acting in one plane in a body.

The conditions of equilibrium of the system of forces considered in the preceding article are

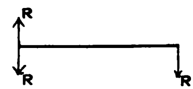
$$R = 0, \text{ and } M = 0.$$

For if M is zero and R is not zero, the resultant will be a single force R.

If R is zero and M is not zero, the resultant will be a couple whose moment is M.

If neither R nor M is zero, the system may be reduced to a

single force. For the couple whose moment is M may be taken as a couple whose forces are R, R and whose arm is M/R, and this couple may be placed so that one of its forces will balance the resultant force R. Hence the resultant of the given system will be a force equal to R.



It follows that the system of forces will not be in equilibrium unless both M and R are zero. Hence the **conditions of equilibrium of a system of forces acting in a plane are**

(1) *The resultant of the system of forces, supposed to act at a point, without change of magnitude or direction, must be zero.*

(2) *The algebraical sum of the moments of forces about any point in the plane of the forces must be zero.*

37. Conditions of equilibrium of constrained bodies.

If a body can only move with a motion of translation in a given direction, *e.g.* a piston in a fixed cylinder, the sufficient and necessary condition of equilibrium is that the *effective component of the forces in that direction must be zero.*

If a point of the body in the plane of the forces is fixed, so that the body can only move with a motion of rotation about that point, *e.g.* a sheet of paper fastened to a board by means of a pin, the sufficient and necessary conditions of equilibrium is that the *algebraical sum of the moments of the forces about the fixed point must be zero.*

In the latter case, there may be a pressure on the fixed point, which would be balanced by the reaction of the fixed point on the body. The pressure on the point will evidently be the resultant of the forces acting on the body, and the reaction of the fixed point on the body will be equal and opposite to this resultant.

EXAMPLES.

1. Forces of 30 and 20 units act on a body in parallel but opposite directions, their lines of action being 6 feet apart. Midway between them a force of 10 units acts parallel to and in the same direction as the force of 20 units. Find the moment of the resultant couple.

The parallel forces 10 and 20 have a resultant 30 acting in the same direction. This resultant and the given force 30, which acts in the opposite direction, form a couple. Hence the resultant of the three forces is a couple.

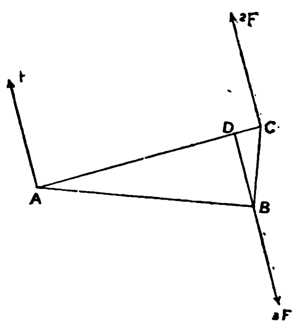
To find the moment of this couple we use the theorem that the algebraical sum of the moments of the forces about any point in the plane of the couple is equal to the moment of the couple. Hence by taking moments about any point in the line of action of the given force 30 we get as the moment of the couple

$$3 \times 10 + 20 \times 6, = 30 + 120, = 150.$$

2. ABC is a triangle right-angled at B, AB being 2 feet long and BC 10 inches. At A a force F is applied at right angles to AC, at C a parallel force $2F$ is applied in the same direction, and at B a parallel force $3F$ in the opposite direction.

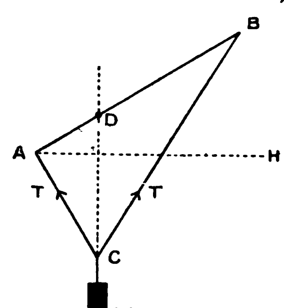
Show that the resultant of the three forces is a couple, and find the moment of the couple taking an inch as the unit of length.

That the resultant is a couple appears by considering that the parallel forces, F at A and $2F$ at C, have a resultant $3F$ acting in the same direction. This force and the force $3F$ at B constitute a couple.



Let the direction of the force $3F$ at B cut AC at D, then BD is the perpendicular from the right angle B upon the hypotenuse AC of the triangle ABC. The moment of the resultant couple is equal to the algebraical sum of the moments of the three forces F, $2F$, $3F$, about any point in the plane. Take moments about the point A. The moment of F at A is zero, the moment of $3F$ at B is $3F \times$ AD, and the moment of $2F$ at C is $2F \times$ AC. Since AB is 24 inches, and BC 10 inches, it follows from geometry that AC is 26 inches, and that AD is 288/13 inches. Hence the moment of the resultant couple is

$$3F \times \text{AD} - 2F \times \text{AC}, = 288F/13.$$



3. A and B are two points at a given distance apart, A below B; the line AB is of given length, and inclined at a given angle to the horizon. A thread of given length has its ends fastened to A and B; a given weight is hung on the thread by a smooth hook. Find the position in which it comes to rest, and the tension of the thread.

Let C be the position of equilibrium of the given weight, W say; CD the vertical through C; AH the horizontal line through A in the plane of the diagram; a the given length AB; and ϕ the given angle BAH.

Since the hook is smooth the tensions in AC and BC, the two parts of the thread, are equal. Denote each of these tensions by T. Then the horizontal components of these two tensions must be equal and opposite, and hence AC and BC must be equally inclined to the horizon, and therefore equally inclined to the vertical line CD. Hence the angles ACD and BCD must be equal. Denote each of these angles by θ.

Since CD bisects the angle, by Euclid VI. 3,

$$\mathrm{AD/AC = BD/BC = (AD+BD)/(AC+BC)}, \\ = a/l,$$

using a well-known formula in Ratio.

Also, by Trigonometry,

$$\mathrm{AD/AC} = \sin \mathrm{ACD}/\sin \mathrm{ADC}. \\ = \sin\theta/\cos\phi.$$

Hence $$\sin\theta/\cos\phi = a/l,$$
giving $$\sin\theta = a\cos\phi/l.$$

This equation determines the position of equilibrium, as it gives the angle which AC and BC make with the vertical.

To find the tension in the thread, we resolve vertically, and equate to zero the algebraical sum of the components. Hence

$$2T\cos\theta - W = 0,$$

from which

$$T = \frac{W}{2}/\cos\theta, = \frac{W}{2}/\sqrt{1-\sin^2\theta},$$

$$= \frac{W}{2}/\sqrt{1-a^2\cos^2\phi/l^2}$$

$$= \frac{Wl}{2}/\sqrt{l^2 - a^2\cos^2\phi}.$$

4. Show that if three forces acting in a plane keep a body in equilibrium, the forces must either be parallel or must meet in a point.

5. Show that a force acting at any point of a rigid body is equivalent to an equal and parallel force acting at any other point and a couple.

6. ABCD is a square, and AC is a diagonal. Two forces of 10 units each act from A to B and from C to D respectively, and a third force of 15 units acts from C to A. Find the resultant.

Ans.—Draw CE perpendicular to AC, and equal to $\frac{2}{3}$ of AD, E being on the same side of AC as D. Then the resultant is a force 15 acting through E parallel to, and in the same direction as, the force 15 in CA.

7. In a weightless straight lever in which the fulcrum is between the power and the weight, show that in the case of equilibrium the power, the weight, and the reaction of the fulcrum form two unlike couples of equal moments.

8. Forces P, 2P, 3P, 4P act along the sides of a square ABCD taken ir

the order AB, BC, CD, DA. Find the magnitude, direction, and line of action of the resultant.

Ans. $2\sqrt{2}$. P acting parallel to CA through a point E in BA produced such that BE $=\frac{7}{2}$BA.

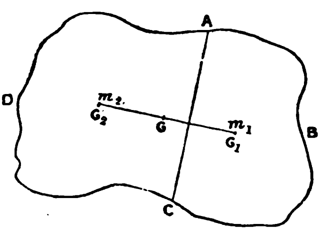
9. ABCDE is a straight line; at the points A, C, E, like parallel forces of 3, 5, 7 units respectively, act perpendicular to AE; and at the points B and D forces of 13 and 2 units respectively act parallel to the former forces, but in the opposite direction. Find the moment of the resultant couple if AB $=$ 2BC $=$ 3CD $=$ 4DE $=$ 12 units of length.

Ans. 65.

Centre of Gravity, Arts. 38 to 40.

38. **Centre of gravity of a body made up of two parts.**

Let ABCD be a body made up of two parts ABC and ACD.



Given the masses, m_1 and m_2 respectively, of the two parts ABC and ACD, and the positions, G_1 and G_2 respectively, of their centres of gravity, it is required to find the position of the centre of gravity G of the whole body.

The weights of the two parts ABC and ACD are parallel forces acting through G_1 and G_2 respectively, and their magnitudes are proportional to m_1 and m_2. The centre of gravity of the whole body, the point G, is the point which is the centre of these two parallel forces; and therefore G will be found by the rule for finding the centre of two parallel forces equal to m_1 and m_2. Hence G is to be found by the equation—

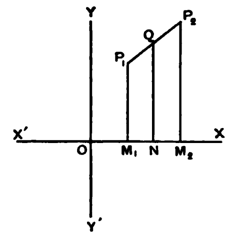
$$GG_1 : GG_2 = m_2 : m_1.$$

39. **Co-ordinates of the centre of gravity of a given system of masses.**

Let $m_1, m_2, m_3, \ldots$ denote the masses, which may be particles or bodies of finite size; and let $P_1, P_2, P_3, \ldots$ be the positions of the masses, if particles, or the positions of their centres of gravity, if bodies of finite size.

First, suppose that $P_1, P_2, P_3, \ldots$ all lie in the same plane.

Take through any point O in the plane in which these points lie any two lines XOX′ and YOY′ at right angles to each other; and draw P_1M_1, P_2M_2, P_3M_3,... perpendiculars from P_1, P_2, P_3,... respectively upon OX. Let x_1, y_1 denote respectively OM_1, P_1M_1; x_2, y_2 denote OM_2, PM_2; and so on. Then the position of the point P_1 with reference to O is determined by the lengths x_1 and y_1, which are called respectively the x and y *co-ordinates*, or the *abscissa* and the *ordinate*, of the point P_1; and similarly the positions of the points P_2, P_3,... are determined by their co-ordinates. Let $(\bar{x}, \bar{y})$, be the co-ordinates of the centre of gravity of the given masses; it is required to find formulæ for $\bar{x}$ and $\bar{y}$ in terms of the given masses and their co-ordinates.



The centre of gravity of the masses m_1 at P_1 and m_2 at P_2 is a point, Q say, in the line P_1P_2, dividing the line P_1P_2 inversely as the masses. Let QN be drawn perpendicular to OX, and let x denote the length of ON.

$$\text{Then}\quad m_1 : m_2 = P_2Q : P_1Q, = M_2N : M_1N \text{ (by Geometry)}$$
$$= (OM_2 - ON) : (ON - OM_1),$$
$$= x_2 - x : x - x_1;$$

from which we get

$$m_1 (x - x_1) = m_2 (x - x_2),$$

giving

$$x (m_1 + m_2) = m_1 x_1 + m_2 x_2.$$

To find the centre of gravity of the three masses m_1, m_2, m_3, we may suppose that the masses m_1 at P_1 and m_2 at P_2 are replaced by a mass $m_1 + m_2$ at Q. The centre of gravity of the three masses, R say, will therefore be the centre of gravity of the mass $(m_1 + m_2)$ at Q and the mass m_3 at P_3. Let x' denote the abscissa of R. Then x' will be found by the formula which has just been found for x. Thus

$$x' \left[(m_1 + m_2) + m_3\right] = (m_1 + m_2) x + m_3 x_3.$$

But

$$(m_1 + m_2) x = m_1 x_1 + m_2 x_2;$$

therefore

$$x' (m_1 + m_2 + m_3) = m_1 x_1 + m_2 x_2 + m_3 x_3.$$

Proceeding in this way we find the centre of gravity of four masses m_1, m_2, m_3, m_4. Let x'' be the abscissa of the centre of gravity of the four masses. Then

$$x''(m_1+m_2+m_3+m_4)=m_1x_1+m_2x_2+m_3x_3+m_4x_4,$$

and in the same way the abscissa, $\bar{x}$, of any number of masses, will be found from the formula

$$\bar{x}(m_1+m_2+m_3+\ldots)=m_1x_1+m_2x_2+m_3x_3+\ldots$$

Using the Σ notation, we may write this formula in the shorter form

$$\bar{x}\Sigma m=\Sigma(mx),$$

from which

$$\bar{x}=\Sigma(mx)/\Sigma m\ldots\ldots\ldots\ldots(1).$$

By similar working to the above we should find the corresponding formula for $\bar{y}$, viz.:—

$$\bar{y}=\Sigma(my)/\Sigma m\ldots\ldots\ldots\ldots(2).$$

Formulæ (1) and (2) completely determine the position of the centre of gravity of the given system of masses when P_1, P_2, P_3,... all lie in the same plane.

If the points P_1, P_2, P_3,... do not all lie in the plane XOY, equations (1) and (2) would be equations determining the position of the point which is the projection of the centre of gravity on the plane XOY. The centre of gravity will be completely determined if its perpendicular distance from the plane XOY were also known. If z_1, z_2, z_3,... denote the perpendicular distances of the masses m_1, m_2, m_3,... respectively from the plane XOY, and $\bar{z}$ the perpendicular distance of the centre of gravity from the same plane, then it may be shown by working similar to that by which the formula for $\bar{x}$ was obtained that

$$\bar{z}=\Sigma(mz)/\Sigma m\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

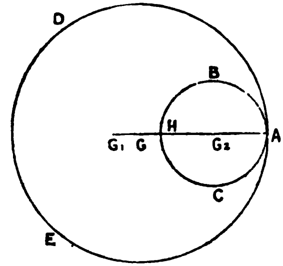
The centre of gravity will therefore be completely determined by formulæ (1), (2), and (3).

These formulæ are also the formulæ for the **centre of a system of parallel forces** m_1, m_2, m_3,... acting at the points P_1, P_2, P_3,... respectively.

Ex. 1.—A circular hole, 6 inches in diameter, is cut out of a circular disc of wood, 15 inches in diameter, close to the edge. Find the centre of gravity of the remaining portion.

Let ADE be the disc, whose centre is G, and let ABHC be the circular

hole, whose centre is G_2, the two circles touching at the point A. We may consider the disc, when entire, as a body made up of two parts, the part ABC and the remaining portion. The centre of gravity of a circle is its centre, and therefore the centres of gravity, G and G_2, of the whole body and the part ABC respectively are known, and it is required to find the centre of gravity of the remaining portion.



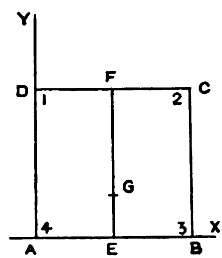
The masses of the part ABC and of the whole disc will be proportional to their areas, and we know from Geometry that the areas of circles are proportional to the squares of their diameters. Hence

$$\text{mass of part cut out : mass of whole disc}$$
$$= 6^2 : 15^2, = 36 : 225;$$

and therefore

$$\text{mass of part cut out : mass of remaining portion}$$
$$= 36 : 225 - 36, = 36 : 189, = 4 : 21.$$

Hence, if G_1 be the centre of gravity of the remaining portion, we have, by Art. 38,
$$GG_1 : GG_2 = 4 : 21.$$
Now GG_2 is the distance between the centre of the disc and the centre of the hole, and is $4\frac{1}{2}$ inches. Hence
$$GG_1 : 4\tfrac{1}{2} = 4 : 21,$$
giving $GG_1 = 6/7$ of an inch, which determines the centre of gravity of the remaining portion.



Ex. 2.—ABCD is a square, and masses of 4 lbs., 3 lbs., 2 lbs., 1 lb. are placed at the angular points A, B, C, D respectively. Find the centre of gravity of the masses.

Take A as origin, AB as axis of x, and AD as axis of y; and let a represent the side of the square.

Then we may exhibit the values of m, x, y, mx, my, for each mass in a table, as follows:—

	m	x	y	mx	my
Mass at A	4	0	0	0	0
.... B	3	a	0	$3a$	0
.... C	2	a	a	$2a$	$2a$
.... D	1	0	a	0	a

By adding the numbers in the column under m, we get

$$\Sigma m = 4+3+2+1, =10.$$

By adding the terms in the columns under mx and my respectively, we get

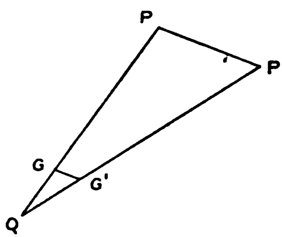
$$\Sigma\ (mx) = 3a+2a, =5a,$$
$$\Sigma\ (my) = 2a+a,\ =3a.$$

Hence the co-ordinates, $\bar{x}$ and $\bar{y}$, of the centre of gravity are—

$$\bar{x} = \Sigma\ (mx)/\Sigma m = 5a/10, = a/2,$$
$$\bar{y} = \Sigma\ (my)/\Sigma m, \quad = 3a/10.$$

Hence we have the following construction for the centre of gravity:—Take E and F the mid points of AB and CD respectively, and measure along EF a length EG equal to 3/10 of EF. Then G is the centre of gravity of the four masses.

Ex. 3.—If a portion m of any mass M is moved to any new position, show that the centre of gravity of the entire mass is thereby moved in a direction parallel to the displacement of the centre of gravity of m and over a distance md/M, where d is the distance between the two positions of the centre of gravity of m.



Let G be the original position of the centre of gravity of the entire mass M, P, P′ the original and displaced positions respectively of the centre of gravity of the mass m. Then PP′ is the displacement of the centre of gravity of m, and is denoted by d.

Let Q be the position of the centre of gravity of the part of the body that remains when the mass m is cut out, and let G′ be the displaced position of the centre of gravity of the entire mass M. Then G is the centre of gravity of the mass m at P and the mass $(M-m)$ at Q, and G′ is the centre of gravity of the mass m at P′ and the mass $(M-m)$ at Q. Hence

$$QG : PG = m : M-m,$$

and $$QG' : P'G' = m : M-m,$$

from which $$QG : PG = QG' : P'G'.$$

Hence, by Euclid VI. 2, GG′, the displacement of the centre of gravity of the entire mass, is parallel to PP′, and the triangles QGG′ and QPP′ are similar.

Also, since

$$QG : PG = m : M-m,$$

therefore $$QG : QG+PG = m : m+M-m,$$

or $$QG : PQ = m : M.$$

Hence $$GG' : PP' = QG : PQ \text{ (Euclid VI. 4).}$$
$$= m : M,$$

from which $$GG' = m \times PP'/M,$$
$$= md/M.$$

This formula for the displacement of the centre of gravity of a mass, consequent on the displacement of a part of the mass, is of importance in the practical determination of the metacentre of a ship.

40. Table of positions of centres of gravity

The following table shows the positions of the centre of gravity of certain surfaces and solids:—

Solid or Surface.	Position of Centre of Gravity.
Sphere,........................	Centre.
Rectangular body (including cube), ...	Middle point of line joining the centres of gravity of two opposite faces.
Cylinder,.......	Middle point of axis.
Pyramid and Cone,........	In line joining the centre of gravity of base to vertex, at distance from vertex equal to 3/4 of the joining line.
Circle,........................	Centre.
Parallelogram (including square and rectangle),...	Point of intersection of diagonals.
Triangle,.....................	In line joining the middle point of a side to opposite angular point, at distance from angular point equal to 2/3 of the joining line.

EXAMPLES V.

(The Answers are given on page 333.)

A.

1. ABDC is a parallelogram, of which AD is a diagonal, and AB is bisected at E. Prove that the resultant of the forces represented by AD and AC is double of the resultant of the forces represented by AE and AC.

2. The sides BC, CA, AB of a triangle ABC are bisected in D, E, and F respectively. Prove that forces represented by AD, BE, and CF would be in equilibrium.

3. Forces of 1, 2, 3, 4, 5, 6, 7, 8 units act from the centre of a regular octagon towards the angular points taken in order. Find the resultant.

4. A force equal to 20 lbwt., acting vertically upwards, is resolved into

two components, of which one is horizontal and equal to 10 lbwt. What is the magnitude and direction of the other component?

5. A force of 50 units acts along a line inclined at an angle of 30° to the horizon; find its horizontal and vertical components.

The force and the two components just found are of course in a vertical plane, find next the components of the force 50 units along two lines in that plane at right angles to each other, and inclined at angles 45° to a horizontal line.

6. Five forces of 18, $12\sqrt{2}$, 50, $3\sqrt{2}$, and $12\sqrt{2}$ units act respectively E., N.E., N., N.W., and W. Find the effective northern component.

7. A sphere, which weighs 100 lbs., rests between two planes which are inclined to the horizon at angles of 30° and 60° respectively. Find the pressure exerted by the sphere on each of the planes.

8. Find the position of equilibrium of a uniform rod, placed on a smooth horizontal rail, with its lower end pressing against a smooth vertical wall parallel to the rail, the length of the rod being 16 feet, and the perpendicular distance of the rail from the wall being 1 foot.

9. A uniform ladder rests in a vertical plane with one extremity on the ground, and the other supported by a vertical wall, both the ground and the wall being smooth. The lower end of the ladder is fastened by a rope to a point at the foot of the wall, the inclination of the ladder to the vertical being 45°. Find the tension of the rope.

10. A square board ABCD, of uniform thickness and density, is capable of turning in its plane, which is vertical, about the corner A. It is held so that AB is vertical by a force acting along BC. Find this force in terms of the weight of the board.

11. A circular lamina, whose radius is 8 inches, weighs 9 oz.; a thin, straight wire as long as the radius of the circle weighs 7 oz.; and the wire is placed so as to be a chord of the circle. Find the distance of the centre of gravity of the whole from the centre of the circle.

12. A cylindrical vessel, open at the top, is made of thin sheet metal of uniform thickness. If the diameter of the vessel is 1 foot, and the height 1 foot, find the height of the centre of gravity from the bottom of the vessel.

If the vessel be half-filled with water, where will the centre of gravity of the water and vessel be, assuming that the weight of the vessel is one-fifth of the weight of the water in it?

13. A hole, 1 foot square, is cut out of a board 4 feet square, the sides of the hole being parallel to the sides of the board, and the middle point of one of them being at the centre of the board. Find the centre of gravity of the remaining part of the board.

14. One of the four parts into which a square is divided by its diagonals is removed. Find the centre of gravity of the remainder.

15. ABC is an equilateral triangle, whose side is 6 inches, and whose

centre of gravity is O. If the triangle OBC be cut out, find the distance from A to the centre of gravity of the remainder.

16. Five masses of 1, 2, 3, 4, 5 lbs. respectively are placed upon a square table. Their distances from one edge of the table are 2, 4, 6, 8, 10 inches, and from the adjacent edge 3, 5, 7, 9, 11 inches respectively. Find the distances of their centre of gravity from the two edges.

17. Weights of 1 lb., 2 lbs., 3 lbs., and 4 lbs. are suspended from a uniform lever 5 feet long at distances of 1 ft., 2 ft., 3 ft., and 4 ft. respectively from one end. If the mass of the lever is 4 lbs., find the position of the point about which it will balance.

18. Masses of 4, 3, 5 pounds are hung at points distant 1, 3, 4 feet respectively from the end A of a uniform lever AB, which is 5 feet long and weighs 6 pounds. Find the distance from the end A of the point about which the lever will balance.

19. A rod AB, 12 feet long, has a weight of 1 lb. suspended from one end A, and when 15 lbs. are suspended from the other end B it balances at a point 3 feet from B, while if 8 lbs. are suspended there it balances about a point 4 feet from B. Find the weight of the rod and the position of its centre of gravity.

20. A rod, 4 feet long, has weights hung from the ends and from each of the foot divisions. The weights taken in order are 8. 4, 7, 10, and 5 lbs. Find where the rod must be supported to balance.

B.

21. Give a geometrical construction for resolving a given force into two rectangular components so that one of the components shall have a given value.

22. A force is given in magnitude and line of action. Give a geometrical construction for resolving it into two other forces which shall be equal to one another, and shall pass through two fixed points.

23. Two equal forces applied at a given point have a resultant given in magnitude and direction. Find the locus of the extremity of the straight line which represents either force.

24. A force F is resolved into two components along two directions inclined to each other at an angle of 60°. The smaller component is a force of 8 units, and makes with F an angle of 45°. Find the force F and the other component, being given $\sin 15° = 13/50$.

25. Prove that if two equal forces, each equal to P, act at a point in directions inclined at an angle $2i$, their resultant is $2P \cos i$.

26. ABCD is a quadrilateral figure; P and Q are the middle points of the opposite sides AB and CD; O is the middle point of PQ. Show that four forces represented by OA, OB, OC, OD respectively are in equilibrium.

27. Two forces are represented completely by two straight lines AB, CD; prove that the resultant is represented in magnitude and direction by 2HK, where H, K are the middle points of AC, BD respectively.

Does the resultant act along HK? Examine the case where ABCD is a parallelogram.

28. A system of forces acts on a particle. Show that if the forces of the system are all increased or all diminished in the same ratio, the resultant is increased or diminished in the same ratio.

29. Show that forces *acting along* and represented by the sides of a polygon taken in order are equivalent to a couple whose moment is equal to twice the area of the polygon.

30. A uniform ladder is placed with one end on the ground and the other leaning against a vertical wall, the ground being rough and the wall smooth. Show that in a position of equilibrium the forces acting on the ladder form two couples of equal moments but of opposite senses.

Also show that the horizontal thrust of the foot of the ladder on the ground bears to the weight of the ladder the ratio of the distance of the foot from the wall to twice the altitude of the upper end.

How does the thrust alter as a man ascends the ladder?

31. If four forces act in one plane on a rigid body, and two of them form a couple, state how to find the resultant of the four forces.

ABCD is a square and AC is the diagonal. Forces of 10, 15, 20, and 15 units act respectively from A to B, B to C, C to A, and D to A. Find the magnitude and line of action of the resultant.

32. ABC is a triangle, and D is the middle point of BC. Forces act from A to B, A to C, A to D, and from B to C, proportional to AB, AC, AD, BC respectively. Find by geometrical construction the magnitude and direction of the resultant.

33. A body is turning about a fixed point under the action of a force whose moment about the fixed point is equal to M. Show that the work done by the force in n revolutions is $2\pi nM$.

34. A piece of thin sheet metal is in the form of a square with equilateral triangles described on three of the sides of the square. If a is the length of a side of the square, prove that the distance of the centre of gravity of the sheet from the centre of the square is $a(5 - \sqrt{3})/22$.

35. On a straight line of length a an isosceles triangle and a square are described, the two figures being equal in area and on opposite sides of the line. How far outside the square is the centre of gravity of the whole?

36. The greatest angle of an isosceles triangle is 120°; from the vertex and the middle points of the equal sides lines are drawn perpendicular to the base. Supposing the perimeter of the triangle and the perpendiculars to be uniform rods of the same material, show that the distance of the centre of gravity of this framework from the vertex is $a(15 + 11\sqrt{3})/48$, where a is the length of the perpendicular from the vertex.

37. Three squares of the same material and same uniform thickness whose areas are A, A, 2A respectively, are placed in one plane so that one corner of each square coincides with a corner of one of the other two, the

three double corners thus being the angular points of a triangle whose sides are edges of the squares. Prove that the centre of gravity of the three squares lies in an edge of the largest square.

38. Two equal masses, each equal to m, are placed at the points A and B respectively. On the straight line AB a segment of a circle is constructed, and a third mass, also equal to m, moves along the circle. What is the locus of the centre of gravity of the three masses?

39. A and B are the centres of gravity of two masses m and n respectively, G is the centre of gravity of the masses m and n, and P is any other point. Show that

$$m \,.\, AP^2+n \,.\, BP^2=m \,.\, AG^2+n \,.\, BG^2+(m+n)PG^2.$$

40. A triangular piece of paper is folded across the line bisecting two sides, the vertex being thus brought to lie in the base. Find the centre of gravity of the paper in this position.

41. Three equal particles are placed anywhere on the three sides of a triangle. If they are moved along those sides in the same sense and over spaces proportional to the sides, show that the centre of gravity of the particles remains at rest.

42. A smooth circular wire is fixed horizontally, and a small weightless ring A can slide on it. The ring is attached to two strings, which pass through holes B and C at the extremities of a diameter, and support weights P and Q respectively. Show that in the position in which the ring is at rest

$$\tan \theta=Q/P,$$

where θ is the angle ABC, and that the pressure of the ring on the wire is

$$\sqrt{P^2+Q^2}.$$

43. A buoy is moored in a stream by two ropes inclined at a right angle, and fastened to points on the bank, the line joining the two points being perpendicular to the direction of the stream. If the force of the stream on the buoy is double the tension of one of the ropes, show that the length of the other rope is double the breadth of the stream.

44. A uniform rod, weight W, which can turn freely about a hinge at its lowest point, is held at an inclination 2θ to the vertical by means of a thread tied to its upper extremity, and fastened to a point vertically above the hinge and at a distance from it equal to the length of the beam. Show that (i) the pressure on the hinge is $W \cos \theta$, (ii) its direction bisects the angle between the rod and the vertical, (iii) the tension of the thread is $W \sin \theta$.

45. The weight of a door is W, the width is a, and the distance between the hinges is b. Assuming that all the weight is borne by the upper hinge, show that the thrust of the door on the lower hinge is $Wa/2b$.

46. ABC is a rigid equilateral triangle, whose weight is neglected; the vertex B is fastened by a hinge to a wall, while the vertex C rests against

the wall at a point vertically below B. If a given weight *W* is hung from A, what are the magnitude and direction of the forces exerted by the triangle on the wall at B and C?

CHAPTER V.—INTEGRATION: MOMENTS OF INERTIA.

Integration, Arts. 41 to 45.

41. Meaning of Integration.

We meet in Dynamics with many problems, for the solution of which it is necessary to find the sum of a series containing an infinite number of terms, the value of each term depending on the value of some varying quantity, and all the terms being infinitesimal in value. The sum is called an *integral*, and the method of finding it is called *integration.*

As an illustration of the process of integration, let us consider the problem of finding the space described in a given time by a body which is moving with a continuously varying velocity. We may conceive that the whole time of motion is divided into a very large number of elements of time, each of which is so small that the velocity during any element will be very nearly uniform. And we may proceed to find the space described on the supposition that the velocity during each element is constant, and equal to its value at the end of the element. The value of the space described, calculated on this supposition, will not represent exactly the space described under the actual conditions of the problem, but will become more and more nearly equal to that space as the elements of time are taken smaller and smaller, or, what is the same thing, as the number of elements is taken greater and greater. When the law according to which the velocity varies is known, the space described during each element of time may be found, and the sum of the spaces described during all the elements may be found. This sum will be the sum of an infinite number of terms, all infinitesimal in value, and the *limiting value of this sum*, as the number of terms becomes greater and

greater, will represent *exactly* the space described by the moving body in the given time.

42. **General formula for an integral.**

Let y represent a quantity whose numerical value depends on the numerical value of another quantity x, the relation between the two quantities being known, so that the numerical value of y is known when that of x is known.

Let a and b be two given values of x, b being greater than a. Divide $(b-a)$ into n equal parts, each equal to h, so that $(b-a)$ is equal to $n\,h$. Then

$$a+h,\ a+2h,\ a+3h,\dots\dots\dots\dots a+n\,h, =b,$$

form a series of values of x. The terms of this series increase by increments, each equal to h, from $a+h$ to b. Denote the corresponding values of y by

$$y_1,\ y_2,\ y_3,\dots\dots\dots\dots y_n$$

respectively. Let each of these values of y be multiplied by h, and let the sum of the products be taken. This sum is

$$(y_1+y_2+y_3+\dots\dots\dots\dots+y_n)h.$$

Using the Σ notation, we can write this in the shorter form, $h\,\Sigma y$. The value of this sum will depend on the relation between x and y and on the value of h. The limit towards which the sum approaches when h is taken smaller and smaller is called **the integral of y between the limits a and b of x.**

Since $h=(b-a)/n$, it follows that as h becomes smaller and smaller, n becomes greater and greater. The process of finding the integral may therefore be described thus:—

Values of x are taken increasing from the value a to the value b by increments each equal to h, and the corresponding values of y are found from the known relation between x and y. Each of these values of y is multiplied by h, and the sum of the products is taken. The limiting value of this sum when h becomes zero and n becomes infinite, is the integral of y between the limits a and b of x.

This is a statement of the general problem, the consideration of which forms the subject of the Integral Calculus. In this book we shall meet only with simple applications of the method of integration. The integrals we shall require will be found by algebraical processes, and without the use of the notation of the Calculus.

We shall require the following formulæ, which are proved in Algebra: *—

Denoting $1+2+3+\dots\dots\dots+n$ by Σn,

and similarly $1^2+2^2+3^2+\dots\dots\dots+n^2$ by Σn^2,

$1^3+2^3+3^3+\dots\dots\dots+n^3$ by Σn^3,

then

$$\Sigma n = n(n+1)/2 \qquad (1).$$

$$\Sigma n^2 = n(n+1)(2n+1)/6 \qquad (2).$$

$$\Sigma n^3 = n^2(n+1)^2/4 \qquad (3).$$

43. Work done by a variable force acting through a given distance.

If a *constant* force F acts on a body through a distance s, the work done is Fs.

If the force F is *variable*, then the work done over the space s must be found by the method of integration. Break up the space s into n equal elements, each equal to h, so that $s = nh$. Let F_1, F_2, F_2,.........F_n denote respectively the values of the variable force at the ends of the first, the second, the third... ......the nth elements. On the supposition that the force is constant over each element, the whole work done over the space s is

$$h(F_1+F_2+F_3+\dots\dots\dots+F_n),$$

and the value of this sum when h is zero and n infinite will be the work done over the space s.

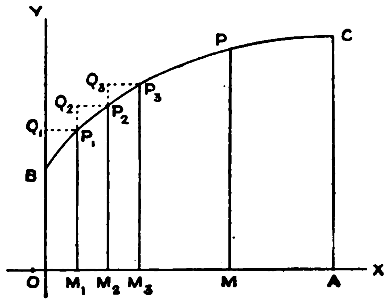
If therefore the law of variation of the force is known, the values of F_1, F_2, F_3,.........F_n will be known, and the value of the integral which represents the amount of work done can be found.

* See Todhunter's *Larger Algebra*, pp. 262 and 263.

44. Geometrical representation of the work done by a variable force.

The work done by a variable force acting on a body through a given distance can be represented by the area of a curve.

Let OX and OY be two lines at right angles through the point O. Let a curve BPC be drawn, such that the abscissa OM of any point P represents the distance described from the initial position of the body to any point of its path, and the ordinate PM the corresponding value of the force. Let OA represent the whole space described, then the work done by the force is represented by the area OACB, bounded by OA, the ordinate AC, the curve CPB, and the ordinate OB.



Let OA be broken up into elements OM_1, M_1M_2, M_2M_3... all equal, and let the ordinates M_1P_1, M_2P_2, M_3P_3,...... be drawn. Though P_1, P_2, P_3,... draw lines parallel to OX, meeting OB, M_1P_1, M_2P_2,... produced in Q_1, Q_2, Q_3,... respectively.

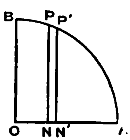
On the supposition that the force over each element is constant, and equal to its value at the end of that element, the work done while the body moves over any element will be equal to the product of the force over that element and the length of the element. Hence the amounts of work done while the body moves over the first, the second, the third, ... elements are $OM_1 \times P_1M_1$, $M_1M_2 \times P_2M_2$, $M_2M_3 \times P_3M_3$,... respectively, that is, are equal to the areas of the rectangles $OM_1P_1Q_1$, $M_1M_2P_2Q_2$, $M_2M_3P_3Q_3$,... respectively. By supposing that the number of elements becomes greater and greater, the lengths of the elements OM_1, M_1M_2, M_2M_3,... will become smaller and smaller, and the rectangles $OM_1P_1Q_1$, &c., will become more and more nearly equal to the elements

OM_1P_1B, $M_1M_2P_2P_1$, $M_2M_3P_3P_2$,... of the curve. But by supposing that the number of elements becomes greater and greater, we are approaching nearer and nearer to the actual conditions of the motion. When the number of elements becomes infinite, the sum of the rectangles will represent the actual amount of work done, and at the same time the sum of the areas of the rectangles will be the area OACB of the curve. Hence the work done is equal to the area of the curve.

45. Centre of gravity of a solid hemisphere.

The centre of gravity of a solid hemisphere may be found by the method of integration.

Let AOB be a quadrant of a circle; then if the area AOB revolve round the line OA, it will generate a solid hemisphere of which OA will be the axis.



Let a denote the radius OA. Divide OA into n equal parts each equal to h, so that $h = a/n$; and let NN′ in the figure be one of these parts. When n is taken infinitely great, h will be an infinitesimal length. In that case the part of the hemisphere enclosed between planes through N and N′ perpendicular to OA may be considered to be an infinitely thin circular disc, whose radius is PN and thickness NN′ or h. If m denote the mass of this element of the sphere, and M the mass of the whole hemisphere, then

$$m/M = \text{volume of disc} / \text{volume of hemisphere},$$
$$= \pi \,.\, \mathrm{PN}^2 \,.\, \mathrm{NN}'/\tfrac{2}{3}\pi a^3, = 3h \,.\, \mathrm{PN}^2/2a^3.$$

Therefore $m = 3hM \,.\, \mathrm{PN}^2/2a^3$.

The centre of gravity of this element will be at N, and therefore the element may be replaced by a particle of mass m placed at the point N. Similarly for all the other elements into which the hemisphere will be divided by planes drawn perpendicular to OA through the points of division on OA. Hence the centre of gravity of the hemisphere will lie on OA.

Let $\bar{x}$ denote the distance of the centre of gravity of the hemisphere from O, and let x denote ON. Then by Art. 39

$$\bar{x} = \Sigma mx/\Sigma m.$$

For the element PNN′P′ we have

$$\begin{aligned} mx &= 3hMx\,.\,\mathrm{PN}^2/2a^3, \\ &= 3hMx(a^2 - x^2)/2a^3, \end{aligned}$$

in which we have substituted for m its value in terms of M, and for PN^2 its equivalent $\mathrm{OP}^2 - \mathrm{ON}^2, = a^2 - x^2$.

By taking the sum of all such products for each of the elements of the hemisphere we obtain the expression for Σmx.

Thus $$\Sigma mx = \Sigma 3hMx(a^2 - x^2)/2a^3.$$

The right-hand side of this equation represents the sum of the series of terms which are obtained from $3hMx(a^2 - x^2)/2a^3$ by putting in succession $x = h$, $x = 2h$, $x = 3h, \ldots\ldots\ldots x = nh$. Taking out the factors common to all the terms of the series we get:

$$\Sigma mx = \frac{3hM}{2a^3}\Sigma x(a^2 - x^2),$$

where $$\begin{aligned} \Sigma x(a^2 - x^2) &= \Sigma(a^2x - x^3) = a^2\Sigma x - \Sigma x^3, \\ &= a^2(h + 2h + 3h + \ldots\ldots + nh) \\ &\quad - \{h^3 + (2h)^3 + (3h)^3 + \ldots\ldots + (nh)^3\}, \\ &= a^2h\,.\,n(n+1)/2 - h^3\,.\,n^2(n+1)^2/4, \end{aligned}$$

using the formulæ (1) and (3) of Art. 42.

Thus

$$\begin{aligned} \Sigma mx &= \frac{3hM}{2a^3}\left\{ a^2h\,.\,n(n+1)/2 - h^3\,.\,n^2(n+1)^2/4 \right\}, \\ &= \frac{3M}{2a^3}\left\{ \frac{1}{2}a^2(nh)^2\left(1 + \frac{1}{n}\right) - \frac{1}{4}(nh)^4\left(1 + \frac{1}{n}\right)^2 \right\} \end{aligned}$$

In the expression on the right-hand side we put $nh = a$, and make n infinitely greater. Then $1/n = 0$, and therefore

$$\begin{aligned} \Sigma mx &= \frac{3M}{2a^3}\left\{ \frac{1}{2}a^4 - \frac{1}{4}a^4 \right\}, \\ &= 3Ma/8. \end{aligned}$$

Also, $$\Sigma m = M.$$

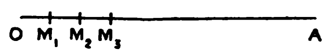
Hence $$\begin{aligned} \bar{x} &= \Sigma mx/\Sigma m, = 3Ma/8M, \\ &= 3a/8. \end{aligned}$$

Hence the centre of gravity of a hemisphere is in the axis at a distance from the centre equal to three-eighths of the radius.

EXAMPLES.

1. A body moves over a distance a, in a straight line, under the action of a force increasing from zero to the value F, the force at any point being proportional to the distance of that point from the initial position of the body. Show that the work done is $Fa/2$.

Let OA represent the distance a, O being the initial position of the body, so that at O the force is zero and at A is F.



Break up OA into n equal parts, each equal to h, so that $nh=a$. Let M_1, M_2, M_3,... be the points of division, so that OM_1, M_1M_2, M_2M_3,... are all equal to h.

Since the force at any point is proportional to the distance of the point from O, it follows that

$$\text{force at } M_1 : \text{force at } A = OM_1 : OA, = h : a.$$

Hence, since the force at A is F, the force at M_1 is Fh/a. Similarly the force at M_2 is $2Fh/a$, at M_3 is $3Fh/a$; and so on.

The work done by the force while the body moves over the element OM_1, on the supposition that the force over the element OM_1 is constant, and equal to its value at M_1, is $(Fh/a)\times h$, or Fh^2/a. Similarly the work done over M_1M_2, on the supposition that the force is constant over M_1M_2, and equal to its value at M_2, is $2Fh^2/a$; the work done over M_2M_3 is $3Fh^2/a$; and so on.

Hence the whole work done on the supposition that over each element the force is constant, and equal to the value of the force at the end of the element, is the sum of the series,

$$Fh^2/a + 2Fh^2/a + 3Fh^2/a + \ldots\ldots\ldots\ldots\ldots + nFh^2/a.$$

The sum of this series is

$$\begin{aligned}
&= (1+2+3+\ldots\ldots+n)\ Fh^2/a, \\
&= n\ (n+1)\ Fh^2/2a, \qquad \text{(by formula 1, Art. 42.)} \\
&= Fn^2h^2\ (1+1/n)\ /2a, \\
&= Fa^2\ (1+1/n)\ /2a, \qquad \text{(since } nh=a\text{)} \\
&= Fa\ (1+1/n)\ /2.
\end{aligned}$$

Now suppose that n becomes greater and greater, then the length of each element of distance becomes smaller and smaller, and the supposition that the force is constant over each element becomes more and more nearly true. Hence the actual amount of work done will be found by supposing that n is equal to infinity. But when n is infinite, $1/n$ is zero, and therefore the work done is $Fa/2$.

Since F increases uniformly with the distance from the initial position, $F/2$ is the average value of the force. Hence the work done is the product of the average value of the force and the distance through which it acts.

2. Prove the result in Ex. 1 by the graphical method.

It easily follows that the curve drawn as in Art. 44 is a straight line through O, and that the work done is represented graphically by the area of a triangle whose base is a, and whose perpendicular is F. Hence the result.

3. Assuming the formula for the length of the circumference of a circle, radius r (Art. 14), prove by integration that the area of the circle is πr^2.

Break up the area of the circle into elements by concentric circles.

4. Assuming that the area of a zone of a sphere varies as its height, show that the centre of gravity of a thin uniform hemispherical shell is in the axis of the hemisphere, at a distance from the centre equal to half the radius.

5. Prove by integration the formula for the volume of a right circular cone. (Art. 14.)

6. Prove by integration that the centre of gravity of a right circular cone lies in the axis of the cone, at a distance from the vertex equal to three-fourths of the length of the axis.

Moments of Inertia, Arts. 46 to 51.

46. Moment of inertia of a body about an axis.

Def.—*If a body be conceived to be broken up into an infinite number of elements, and if the products of the mass of each element and the square of its perpendicular distance from a given straight line be formed, the sum of the products for all the elements of the body is the moment of inertia of the body about the given straight line.*

The word "body" in this definition is to be taken to include a system of particles which are not continuous. The given straight line in the definition is usually referred to as "the axis," about which the moment of inertia is taken.

In the case in which the body consists of a finite number of particles, the moment of inertia about an axis will be the sum of a finite number of terms, and may be found by the rules of elementary Algebra. When the body is a finite mass, the problem of finding the moment of inertia will involve the process of integration.

When the body is in the form of a very thin plate or lamina

of uniform density and thickness, we may look upon the plate as a surface—that is, as a body having length and breadth only —but at the same time differing from a surface in having mass. And, in finding its moment of inertia, we may conceive that the lamina is broken up into elements, all lying in the same plane, the mass of each of which is proportional to its area. Thus we may use the expression, "moment of inertia of a circle," in the same way as we use the expression "centre of gravity of a circle."

In some cases it is convenient to speak of the moment of inertia of the *area* of a plane figure, such as a circle or a rectangle. When we use this language, we refer to the sum of the products of the elements of an *area* about a specified axis. This may be found from the moment of inertia of the *mass* of a uniform lamina, which is in the shape of the area, by supposing that all letters used to denote masses denote areas. For the masses of the elements of a uniform lamina are proportional to the areas of the surfaces of the elements.

It is usual to express the moment of inertia of a body in the form of the product of the mass of the whole body and the square of a length. If M denote the mass of a body, whose moment of inertia about a specified axis has been found, and if k denote a length such that Mk^2 is the moment of inertia, then k is called the **radius of gyration** of the body about the specified axis. Hence the definition :—

Def.—*If M denote the mass of a body, and k denote a length such that Mk^2 is the moment of inertia of the body about a given axis, then k is called the radius of gyration of the body about that axis.*

For example, if m denote the mass of each of four equal particles, placed one at each of the corners of a square, ABCD, whose side is $2a$, the moment of inertia of the four masses about an axis through an angular point A, perpendicular to the plane of the square, is—

$$m \,.\, AB^2 + m \,.\, AC^2 + m \,.\, AD^2,$$
$$= m\,(2a)^2 + m \,.\, (2a\sqrt{2})^2 + m\,(2a)^2, = 16ma^2.$$

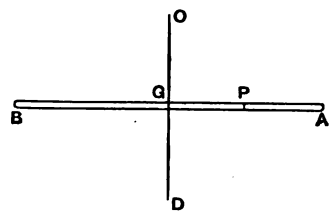
Since $16ma^2 = 4m \times 4a^2$, it follows that the square of the radius

of gyration about the axis in question is $4a^2$, and the length of the radius of gyration is $2a$.

In the following Articles we shall use M.I. as an abbreviation for moment of inertia.

47. Moment of inertia of a uniform rod about an axis through its middle point perpendicular to its length.

Let AB be a uniform rod of length $2a$ and mass M. It is required to find its M.I. about CD, an axis perpendicular to its length through its middle point G.



Break up each of the halves GA and GB of the rod into n equal elements; take the sum of the M.I.'s of the elements about the axis CD; and find the value of the sum when n becomes infinite.

Let $a/n = h$, so that h is the length of each infinitesimal element of the rod. Let P be any element, which we may consider to be a particle, and let x denote the length of GP. If m denote the mass of this element,

$$m/M = h/2a;$$

and therefore $$m = Mh/2a.$$

Hence the M.I. about CD of the element at P

$$= (Mh/2a) \,.\, \mathrm{GP}^2, = (Mh/2a) \,.\, x^2,$$
$$= Mhx^2/2a.$$

In this expression we put $x = h$, $x = 2h$, $x = 3h, \ldots$ successively, and add all the results. The value of the sum when h is zero or n infinite will be the M.I. of the half GA of the rod about the axis CD. Hence the moment of inertia of the part GA

$$= \Sigma\, Mhx^2/2a,$$
$$= (Mh^3/2a) \,.\, (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2),$$
$$= (Mh^3/2a) \,.\, n\,(n+1)\,(2n+1)/6, \qquad \text{(Art. 42)}$$
$$= (M/12a) \,.\, (nh)^3 \,.\, (1 + 1/n)\,(2 + 1/n),$$
$$= (M/12a) \,.\, a^3 \,.\, 2, = Ma^2/6,$$

putting $nh = a$, and $1/n = 0$.

Similarly the M.I. of the half GB of the rod is $Ma^2/6$, and therefore the M.I. of the whole rod is $2Ma^2/6$, or $Ma^2/3$.

If k denote the radius of gyration, then

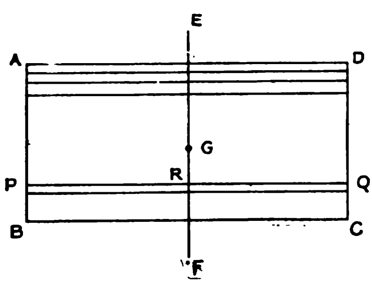
$$Mk^2 = Ma^2/3;$$

therefore $$k = a/\sqrt{3}.$$

48. **Moment of inertia of a rectangle about an axis in its plane through the point of intersection of the diagonals perpendicular to a side.**

By rectangle we mean a very thin rectangular lamina of uniform density and thickness. The masses of parts of this lamina will be proportional to the areas of the parts, and we may therefore deduce the M.I. of the *area* of the rectangle from the M.I. of the *mass* of the rectangle.

Let ABCD be the rectangle, whose sides AD and BC are each equal to $2a$, and whose mass is M. Let G be the point of intersection of the diagonals, and EGF the line through G perpendicular to the sides AD and BC.



Let the rectangle be divided up into infinitely thin strips by straight lines parallel to AD, and let PQ be any one of these strips. Then PQ may be treated as a rod of infinitesimal section. Let EF cut PQ in R, then R is the middle point of PQ, and EF is therefore an axis through the middle point R of the rod PQ. Hence if m denote the mass of the element PQ, the M.I. of this element about PQ is $m \, . \, a^2/3$. Hence the M.I. of the rectangle about EF

$$= (a^2/3) \times \text{sum of the masses of the elements}$$
$$= Ma^2/3.$$

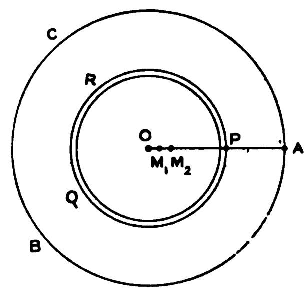
Hence the formula for the M.I. of a rectangle about an axis joining the middle points of two sides, whose lengths are each equal to $2a$, is the same as the formula for the M.I. of a rod of

equal mass and of length $2a$. Also, if k denote the length of the radius of gyration, k is equal to $a/\sqrt{3}$.

If A denote the *area* of the rectangle, then the M.I. of the area of the rectangle about the axis EF is $A \cdot a^2/3$.

49. Moment of inertia of a circle about an axis through its centre perpendicular to its plane.

Let ABC be a circular lamina whose radius is a, and mass M. Take a radius OA, and divide it at $M_1, M_2, M_3, \ldots$ into n equal and infinitesimal parts OM_1, M_1M_2, $M_2M_3, \ldots$ each equal to h, so that $nh=a$. With centre O and radii OM_1, OM_2, $OM_3, \ldots$ respectively let circles be described. These concentric circles will divide the circle ABC into rings, each of breadth h. Let PQR be one of these rings; let OP, the radius of the ring, be denoted by x, and let m denote its mass.



Then

$$m/M = \text{area of PQR} : \text{area of circle ABC},$$
$$= 2\pi x \cdot h : \pi a^2, \qquad \text{(Art. 14)},$$

and therefore $m = 2Mhx/a^2$.

The M.I. of this ring about an axis through O perpendicular to the plane of the circle

$$= m \times OP^2, = mx^2, = 2Mhx^3/a^2.$$

The M.I. of the whole circle about this axis

$$= \Sigma\, 2Mhx^3/a^2, = (2Mh^4/a^2) \cdot (1^3 + 2^3 + 3^3 + \ldots\ldots + n^3),$$
$$= (2M/a^2) \cdot (nh)^4 \cdot (1 + 1/n)^2/4,$$
$$= Ma^2/2, \text{ putting } nh = a \text{ and } 1/n = 0.$$

If k denote the radius of gyration of the circle about the same axis, then k is equal to $a/\sqrt{2}$.

The M.I. of the *area* of the circle about the same axis is $A \cdot a^2/2$, where A denotes the area of the circle.

Cor.—The M.I. of a right circular cylinder, whose mass is

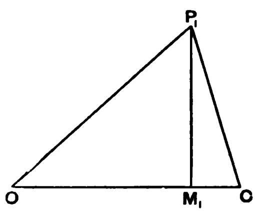
M, and the radius of whose cross section is *a*, about its axis is $Ma^2/2$. This follows immediately from the formula for the M.I. of a circle by breaking up the mass of the cylinder into elements by planes perpendicular to the axis.

50. Propositions regarding Moments of Inertia.

Prop. I.—*If M denote the mass of a body, I the M.I. about any axis through the centre of gravity of the body, and I′ the M.I. about a parallel axis at a distance h from the centre of gravity, then*

$$I' = I + Mh^2.$$

Conceive the body to be broken up into an infinite number of masses $m_1, m_2, m_3, \ldots$ whose positions are $P_1, P_2, P_3, \ldots$



Let the plane of the paper be the plane through P_1 perpendicular to the parallel axes, and let this plane cut the axis through the centre of gravity in O, and the parallel axis in C. Then $OC = h$; and OP_1 and CP_1 are the perpendiculars from P_1 upon the parallel axes through O and C respectively. Draw P_1M_1 perpendicular to OC, and let $OM_1 = x_1$. Then

$$CP_1^2 = OP_1^2 + OC^2 - 2OC \, . \, OM_1 \text{ (Euclid II. 13)},$$
$$= OP_1^2 + h^2 - 2hx_1.$$

Therefore

$$m_1 \, . \, CP_1^2 = m_1 \, . \, OP_1^2 + m_1 \, . \, h^2 - 2h \, . \, m_1x_1.$$

Write down similar equations for each of the masses m_2, $m_3, \ldots$, and add all the equations together.

The result is

$$\Sigma m \, . \, CP^2 = \Sigma m \, . \, OP^2 + h^2\Sigma m - 2h\Sigma mx, \qquad \text{(A)}$$

where $\Sigma m \, . \, CP^2 = m_1 \, . \, CP_1^2 +$ similar terms for $m_2, m_3 \ldots\ldots$,

$$= I;$$

$$\Sigma m \, . \, OP^2 = m_1 \, . \, OP_1^2 + \text{similar terms for } m_2, m_3 \ldots\ldots,$$
$$= I';$$
$$\Sigma m = m_1 + m_2 + m_3 + \ldots\ldots,$$
$$= M;$$

and $\Sigma mx = m_1x_1 + m_2x_2 + m_3x_3 + \ldots\ldots$

$=$ sum of products of each of the masses m_1, m_2, m_3,... by its perpendicular distance from the plane through O perpendicular to OC,

$= M\bar{x}$, where $\bar{x}$ is the distance of the centre of gravity of the body from the same plane (by Art. 39)

$= 0$, since $\bar{x} = 0$, the centre of gravity lying in this plane.

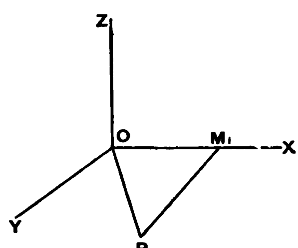
Hence the equation (A) becomes

$$I' = I + Mh^2.$$

Prop. II.—*If A and B denote respectively the M.I.'s of a lamina about two axes at right angles, OX and OY in the plane of the lamina, and if C denote the M.I. about an axis OZ through O perpendicular to the plane of the lamina, then*

$$C = A + B.$$

As in Prop. I., break up the lamina into infinitesimal elements, whose masses are m_1, m_2, m_3,... respectively, and let P_1, P_2, P_3,... be the positions of the masses.



Draw P_1M_1, P_2M_2, P_3M_3,... perpendicular to OX. Then, remembering that the body is a lamina, that is, that all the masses lie in the plane XOY,

$$A = \text{M.I. about OX},$$
$$= m_1 . P_1M_1^2 + m_2 . P_2M_2^2 + m_3 . P_3M_3^2 + \ldots\ldots,$$

and

$$B = \text{M.I. about OY},$$
$$= m_1 . OM_1^2 + m_2 . OM_2^2 + m_3 . OM_3^2 + \ldots\ldots.$$

Hence by addition

$$A + B = m_1(PM_1^2 + OM_1^2) + m_2(P_2M_2^2 + OM_2^2) + m_3(P_3M_3^2 + OM_3^2) + \ldots\ldots,$$
$$= m_1 . OP_1^2 + m_2 . OP_2^2 + m_3 . OP_3^2 + \ldots\ldots,$$
$$= C,$$

which proves the proposition.

Cor. 1.—If k denote the radius of gyration of any body about an axis through the centre of gravity, and k_1 denote the radius of gyration about a parallel axis at a distance h from the centre of gravity, then

$$k_1^2 = k^2 + h^2.$$

For in Prop. I.,

$$I = Mk^2,\ I' = Mk_1^2,$$

and therefore $$Mk_1^2 = Mk^2 + Mh^2,$$

or $$k_1^2 = k^2 + h^2.$$

Cor. 2.—If k_1 and k_2 denote the radii of gyration of a *lamina* about two intersecting axes in the plane of the lamina, and if k_3 denotes the radius of gyration about an axis perpendicular to the plane of the lamina through the point of intersection of the first two axes, then

$$k_3^2 = k_1^2 + k_2^2.$$

This follows from Prop. II. For $A = Mk_1^2$, $B = Mk_2^2$, and $C = Mk_3^2$. Hence, since $C = A + B$,

$$Mk_3^2 = Mk_1^2 + Mk_2^2,$$

giving $$k_3^2 = k_1^2 + k_2^2.$$

51. Applications of the Propositions of Article 50.

We shall apply the propositions of the preceding Article to find certain moments of inertia.

(i.) **Rod about an axis through one end perpendicular to its length.**

As in Art. 47, let $2a$ denote the length of the rod, and M its mass. Let I denote the M.I. of the rod about an axis through its middle point perpendicular to its length, and let I' denote the M.I. about a parallel axis through an end of the rod. Then $I = Ma^2/3$ (Art. 47), and $h = a$.

Thus
$$I' = I + Mh^2 \quad \text{(Prop. I. Art. 50)}$$
$$= Ma^2/3 + Ma^2,$$
$$= 4Ma^2/3.$$

(ii.) **Circle about a diameter.**

As in Art. 49, let M denote the mass of the circle, and a the radius. Let C denote the M.I. of the circle about an axis through the centre perpendicular to the plane of the circle, and let A and B denote the M.I.'s about two diameters of the circle at right angles. Then $C = Ma^2/2$, and it is obvious that $A = B$. Hence, by Prop. II. Art. 50,
$$C = A + B, = 2A;$$
therefore
$$Ma^2/2 = 2A,$$
from which
$$A = Ma^2/4.$$

(iii.) **Circle about an axis in its plane at a given distance from its centre.**

Let M denote the mass of the circle, a its radius, and c the given distance. Then, by Prop. I. Art. 50,
$$I' = I + Mh^2 = Ma^2/4 + Mc^2,$$
$$= M(a^2/4 + c^2).$$

(iv.) **Rectangle about an axis through its centre of gravity perpendicular to its plane.**

As in Art. 48, let M denote the mass of the rectangle, and let $2a$ and $2b$ denote respectively the lengths of two adjacent sides. Take A and B to denote the moments of inertia about axes through the centre of gravity perpendicular to the sides $2a$ and $2b$ respectively, and C to denote the M.I. about an axis through the same point perpendicular to the plane of the rectangle. Then, by Prop. II. Art. 50,
$$C = A + B, = Ma^2/3 + Mb^2/3 \quad \text{(Art. 48)},$$
$$= M(a^2 + b^2)/3.$$

This formula also gives the M.I. of a rectangular solid, mass M, edges $2a$, $2b$, $2c$ about an axis through its centre of gravity perpendicular to the plane of the edges $2a$ and $2b$. This may be proved by breaking up the body into thin slices.

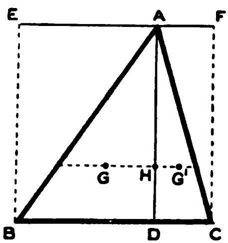
(v.) **Rectangle about a side.**

Let M denote the mass of the rectangle, and $2b$ the length of the sides perpendicular to the side about which the M.I. is required. Let I' denote the M.I. required, and I the M.I. about a parallel axis through the centre of gravity. Then $I = Mb^2/3$, and, by Prop. I. Art. 50,

$$I' = I + Mh^2, = Mb^2/3 + Mb^2,$$
$$= 4Mb^2/3.$$

(vi.) **Triangle about a side.**

Let ABC be a triangle, mass M, and let h denote the length of AD, the perpendicular from the angular point A upon the side BC. The M.I. of the triangle ABC about the side BC shall be equal to $Mh^2/6$.



Draw BE and CF perpendicular to BC, and through A draw EAF parallel to BC. Then the figures EBCF, EBDA, and ADCF are rectangles. The diagonal AB divides the rectangle EBAD into two triangles, EAB and ABD, which are equal in all respects. Similarly the triangles ADC and ACF are equal in all respects.

Let M_1 and M_2 denote respectively the masses of the triangles ABD and ADC, and let k_1 and k_2 denote respectively the radii of gyration of these triangles about the axis BC. Let G and G′ denote respectively the centres of gravity of these triangles; then GG′ will be parallel to BC, and will cut AD in a point H such that DH $= h/3$.

By (v.) of this Article, the M.I. of the *rectangle* AEBD about AE is $2M_1 . h^2/3$. It is also equal to the sum of the M.I.'s about the same axis of the triangles AEB and ABD. Now, since the triangles AEB and ABD are congruent,

M.I. of triangle AEB about EA = M.I. of triangle ABD about BD $= M_1k_1^2$;

and, by Cor. 1 of Article 50, if K denote the radius of gyration of the triangle ABD about GG′,

$$K^2 = k_1^2 - \mathrm{HD}^2, = k_1^2 - (h/3)^2,$$
$$= k_1^2 - h^2/9.$$

Thus the M.I. of the triangle ABD about AE

$$= M_1(K^2 + \mathrm{AH}^2), = M_1(k_1^2 - h^2/9 + 4h^2/9),$$
$$= M_1(k_1^2 + h^2/3).$$

Hence the sum of the M.I.'s of the triangles AEB and ABD about AE

$$= M_1k_1^2 + M_1(k_1^2 + h^2/3),$$
$$= M_1(2k_1^2 + h^2/3).$$

Hence $$M_1(2k_1^2 + h^2/3) = 2M_1h^2/3.$$
from which $$M_1k_1^2 = M_1h^2/6.$$

Similarly the M.I. of the triangle ADC about BC, $M_2k_2^2$, is equal to $M_2h^2/6$. Therefore the M.I. of the whole triangle ABC about BC.

$$= M_1k_1^2 + M_2k_2^2 = M_1h^2/6 + M_2h^2/6,$$
$$= (M_1 + M_2)h^2/6$$
$$= Mh^2/6.$$

EXAMPLES.

1. Uniform rod, length $2a$, about an axis, perpendicular to its length, through a point at a distance c from the centre of gravity—

$$k^2 = a^2/3 + c^2.$$

2. Rectangle, sides $2a$, $2b$, about an axis through an angular point perpendicular to its plane—

$$k^2 = 4(a^2 + b^2)/3.$$

3. Same rectangle, about an axis, parallel to sides $2a$, at a distance c from the centre of gravity—

$$k^2 = b^2/3 + c^2.$$

4. Fine wire bent into the form of a circle, radius a, about an axis through any point of the wire perpendicular to its plane—

$$k^2 = 2a^2.$$

5. Circle, radius a, about an axis perpendicular to its plane through any point of its circumference—

$$k^2 = 3a^2/2.$$

6. Circle, radius a, about an axis in its plane at a distance c from the centre—

$$k^2 = a^2/4 + c^2.$$

7. Square, side $2a$, about a diagonal—

$$k^2=a^2/3.$$

8. Lamina bounded by two concentric circles, radii a and a', about an axis through the common centre perpendicular to the plane of the lamina—

$$k^2=(a^2+a'^2)/2.$$

9. Right-angled isosceles triangle, length of equal sides a, about an axis through its centre of gravity perpendicular to its plane—

$$k^2=a^2/9.$$

10. Any triangle about an axis through an angular point parallel to the opposite side, distance of the angular point from the opposite side being h—

$$k^2=h^2/2.$$

11. Sphere, radius a, about a diameter—

$$k^2=2a^2/5.$$

Chapter VI.—FLUID PRESSURE.

Stress, Arts. 52 and 53.

52. Stress upon a given section of a body at rest under given external forces.

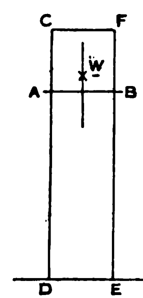
In considering the forces which maintain the equilibrium of a body, whether it be a solid or a mass of fluid, it is not necessary to take account of the internal forces in the body. These forces consist of an infinite number of pairs of forces acting between particle and particle of the body, and each of these pairs of forces forms a balancing system. For by the third law of motion the mutual action between two particles consists of two equal forces acting in opposite directions in the same straight line. It follows, therefore, that the whole system of internal forces will form a balancing system of forces, which may be left out of account in considering the equilibrium of the body as a whole.

Let us now conceive that a surface is drawn in the body, dividing the body into two parts, which may be denoted by A and B respectively. Each of these parts is an example of a body at rest, and we may consider the forces which maintain it in equilibrium. Thus the part A is at rest under the action of

the given external forces acting on it and of the force which the part B exerts on it. Similarly the part B is in equilibrium under the action of the external forces acting on it and of the reaction on it of the part A. It follows that, in considering the equilibrium of the part A, the force exerted on that part by the other part B is to be treated as one of the forces which maintain A in equilibrium. From this it is evident that the action of one part B on the other part A is the force which balances the resultant of the external forces acting on A; and similarly that the reaction of A upon B is the force which balances the resultant of the external forces acting on B.

The mutual action and reaction between two parts of a body is called the *stress upon the section of the body made by the imaginary surface separating the two parts.* We have seen that this stress consists of two equal and opposite forces, the two *aspects* of the stress, and that these are the forces which balance the external forces acting on the two parts respectively.

Ex. 1.—A pillar of stone in the form of a circular cylinder stands on the ground with its axis vertical. Find the intensity of the stress across a horizontal section of the pillar at a depth of 3 feet below the top, if a cubic foot of the stone weighs 200 lbs.



Let CDEF be pillar, and let AB be a section at a depth 3 feet below the top CF.

The part CABF of the pillar is in equilibrium under the action of its weight and the force exerted on it by the lower part ADEB of the pillar. The latter force is one aspect of the stress across the plane AB. Hence the stress across AB will be a *pressure* equal to the weight of the part CABF. This stress will evidently be uniformly distributed over the area of the plane AB, and the *intensity of the stress* or the *force per unit of area* will be the weight, W say, of the part CABF, divided by the number of units of area, A say, in the section AB. Thus the intensity of stress per square foot on AB

$$= W/A, = 200 \times 3 \times A/A, = 600 \text{ lbwt.}$$

Ex. 2.—If the cylinder in Ex. 1 is suspended by a chain attached to the centre of the end section CF, and hangs with its axis vertical, find the intensity of stress across a horizontal section, 12 feet above the lower end of the cylinder.

Take AB in above figure to represent the section in question; let W' de-

note the weight of the lower part ADEB of the cylinder, and A the area of the section. Then the lower part ADEB is in equilibrium under the action of its weight W' and the action of the upper part upon it. These forces must be equal and opposite, and therefore the action of the upper part on the lower part is a *tension* equal to W′. Hence the intensity of the stress on the section or the force per unit of area is

$$= W'/A, = 200 \times 12 \times A/A, = 2400 \text{ lbwt.}$$

53. Stress in a fluid at rest.

Since fluids cannot exert nor transmit a tension, it follows that the stress upon any section of a mass of fluid must be a force of the nature of a *pressure.*

Also, the pressure upon any section of a fluid at rest must be in a direction perpendicular to the section. For a fluid, whether it be perfect or viscous, yields to the action of the smallest force tending to make one part of it slide over another part. It follows that in a fluid at rest there can be no force tangential to the surface separating two parts of the fluid. Thus we arrive at the result that *the stress between two parts of a fluid at rest is a pressure which is at every point perpendicular to the surface separating the two parts.*

The same reasoning will apply in considering the pressure of a fluid on a rigid surface, *e.g.* the pressure of water on the sides of the vessel which contains it, or the pressure of a fluid on the surface of another fluid, *e.g.* the pressure of the atmosphere on the surface of a sheet of water. It follows as before that the pressure of the fluid on the surface, and the reaction of the surface on the fluid, must act perpendicular to the surface.

Hence we have the following as the **fundamental property of a fluid at rest**:—

The pressure of a fluid on any surface with which it is in contact is perpendicular to the surface at every point.

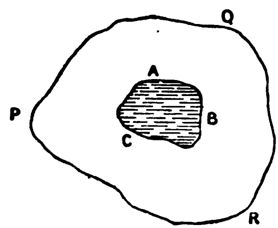
In this statement the word "surface" is to be taken to include (i) an imaginary surface drawn through the mass of the fluid, (ii) a rigid surface exposed to the pressure of the fluid, *e.g.* the sides of a vessel containing liquid, or (iii) the surface of another fluid, *e.g.*, the surface of a bubble of air in water.

Equilibrium of Fluids, Arts. 54 to 60.

54. **Application of the principles of Statics to Fluids at rest.**

In a fluid at rest every part of the fluid is in equilibrium. We may, therefore, apply the principles of Statics in considering the forces which keep a definite mass of the fluid at rest. In general these forces are (i) the pressures exerted on the mass under consideration by the surrounding fluid, (ii) the reaction of rigid surfaces on the mass, and (iii) the weight of the mass. The internal forces in the mass itself are left out of account, as they form a balancing system of forces.

If, for example, PQR is a mass of fluid, we may consider the forces which keep the portion ABC of the fluid in equilibrium. The forces acting on ABC are (i) the pressures exerted on ABC by the surrounding fluid, forces acting perpendicular to the surface of ABC; (ii) the reaction of any surface with which ABC may be in contact; and (iii) the weight of the mass ABC.



55. **Intensity of pressure of a fluid at a point.**

When a surface is exposed to the pressure of a fluid, we may conceive that the area of the surface is divided into an infinite number of infinitesimal elements, all equal in area. If the force exerted on each of these elements is the same, the pressure is said to be *uniform* over the surface; but if the force on each element is not the same, the pressure on the surface is said to be *variable.*

When a surface is exposed to a *uniform* fluid pressure, the **intensity of pressure** at any point of the surface **is the force exerted on unit of area** of the surface.

To measure the intensity of pressure at a given point of a surface, exposed to a pressure which is *not uniform*, we must apply the usual method by which a variable quantity is measured. Take an element of the surface enclosing the point,

the area, a say, of the element being so small that the pressure, F say, exerted on the element may be considered to be uniform. Take a unit of area, imagine that it is broken up into elements of area a, and that each of these elements is pressed with a force F. Then the whole force which would be exerted on this unit of area is the measure of the intensity of pressure at the given point of the surface.

To measure pressure at a point in the interior of a mass of fluid, we imagine a plane drawn in the fluid through the point. The intensity of the stress upon this plane at the point is the measure of the intensity of pressure at the point.

The student will notice that we speak of the intensity of pressure *at* a point, and not *on* a point. The pressure *on* a point is zero, but the intensity of pressure *at* a point is the force that would be exerted on unit of area if over a unit of area the intensity of the pressure were the same as at the point.

The word *pressure* is frequently used in Hydrostatics as an abbreviation for the expression *intensity of pressure.*

The number which measures pressure intensity will of course depend on the units of area and force. It is usually convenient to express pressures in gravitational units. Thus pressures may be expressed in pounds weight per square inch or per square foot, or in grammes weight per square centimetre.

The absolute unit of pressure intensity in the C.G.S. system is usually used in stating experimental results in physics. It is the pressure of one dyne per square centimetre.

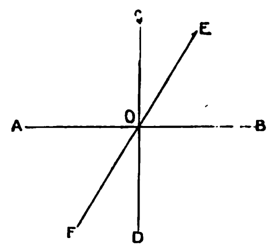
The surface of a boiler containing steam is an example of a surface exposed to a pressure, which, without sensible error, may be considered to be *uniform.* The pressure intensity at any point of the surface of the boiler is usually expressed in pounds weight per square inch, and is measured by a pressure-gauge.

In the case of a ship floating in water, the part of the sides in contact with the water is an example of a surface exposed to a fluid pressure which is *not uniform.* In this case the intensity of pressure at a given point of the sides might be found by measuring by means of a delicate pressure-gauge the force exerted by the water on a very small area enclosing the point. Thus the pressure over an area of 1/16 of a square inch enclosing the point will be sensibly uniform. If, therefore, the force exerted by the water on this area, expressed in pounds weight, is measured by the gauge,

the product of the number registered by the gauge and the number 16 will be the measure of the pressure intensity at the given point. For this is the pressure which would be exerted on a square inch, if over a square inch the pressure were uniform and equal in intensity to the pressure at the given point.

56. Equality of pressure in all directions.

At a point in the interior of a mass of fluid the intensity of the stress is the same in all directions.



This proposition can be deduced mathematically from the fundamental property of a fluid (see p. 340), but it is here stated as an experimental result.

Thus if O is a point in the interior of a mass of fluid, and planes AB, CD, EF be drawn in different directions through O, the proposition states that the pressure on very small equal areas of the planes AB, CD, EF enclosing the point O will be equal.

That at a point in a fluid there is a stress in all directions may be illustrated by means of the following experiment:—

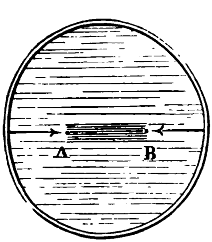
Soak a piece of thin cardboard in water, and press it with the hand against one end of a glass cylinder, which is open at both ends. On immersing in water the end against which the cardboard is pressed, and then withdrawing the hand, it will be found that the water presses the cardboard tightly against the end of the cylinder, and that no water enters the cylinder. This result will follow whether the cylinder is held vertically, or in any position inclined to the vertical, and clearly shows that the water exerts at a point a pressure in all directions. Now hold the cylinder vertically, and pour water gently into it. It will be found that as long as the level of the water inside the cylinder is lower than the level outside, the cardboard remains fast, and that when the water inside has risen to the level of the water outside, the cardboard floats away. This shows that the upward pressure of the water on the card is equal to the weight of the column of water inside the cylinder.

57. Laws of variation of pressure in fluids at rest. Fluids, being forms of matter, are acted on by the force of gravity, and therefore have weight.

In a fluid the pressure is the same in all directions at a given point, but varies from point to point. We shall prove that the variation is due to the force of gravity, and that if we neglect the weight of the fluid, the pressure will be the same at every point. We shall prove **two propositions which are true for all fluids.**

Prop. I. *The pressure is the same at all points in the same level.*

Let A and B be two points in the same level; then shall the intensity of pressure at A be equal to the intensity of pressure at B.



Let p and p' denote respectively the intensities of pressure at A and B. Imagine a circular cylinder of very small section to be described round AB as axis, the ends of this cylinder being perpendicular to the axis, and let a represent the area of its section.

This cylinder of fluid is at rest under the action of (1) the pressures of the fluid on the end sections at A and B of the cylinder, (2) the pressure of the surrounding fluid on the curved surface of the cylinder, (3) the weight of the fluid in the cylinder.

Since the pressure of a fluid in contact with a surface is perpendicular to the surface, the pressure on the end at A is perpendicular to that end, and therefore acts in the direction from A to B. Similarly the pressure on the end at B is a force acting from B to A.

The pressure of the fluid on an element of the curved surface is perpendicular to that element, and is therefore perpendicular to the axis AB. Hence the resultant of the fluid pressure on the curved surface acts perpendicular to the axis AB, and has no component parallel to AB.

The weight of the cylinder of fluid is a force acting vertically downwards through the centre of gravity of the cylinder of

fluid, and therefore this force also acts perpendicular to the axis AB, and has no component parallel to AB.

Hence the pressures on the end sections at A and B are the only forces acting on the cylinder of fluid in the direction of the axis, and these forces must therefore be in equilibrium. It follows that the pressures on the ends at A and B must be equal.

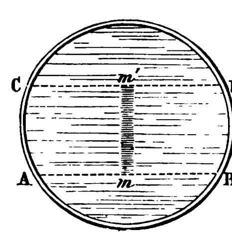
Since the area, a, of the section is very small, the pressure on the end at A will be uniform, and equal in intensity to p. Hence the the whole pressure on the end A is pa, and similarly the whole pressure on the end B will be $p'a$. Hence

$$pa = p'a,$$

from which $p = p'$, which proves the proposition.

Prop. II. *The difference between the intensities of pressure at two points at different levels is equal to the weight of a vertical column of the fluid, of unit sectional area, extending from the level of the lower point to the level of the upper point.*

Let AmB and Cm'D be two horizontal lines in a mass of fluid. Then the pressure at all points of the horizontal plane through AB is the same; and similarly the pressure at all points of the horizontal plane through CD is the same. We may, therefore, in proving the proposition, suppose that the two points, the pressures at which are under consideration, are in the same vertical line.



Let m and m' be two points in the same vertical line, and let us consider the equilibrium of a cylinder of fluid between m and m', the section of this cylinder being of unit area, and the ends being horizontal.

The forces acting on this column of fluid are (1) the pressures of the fluid on the ends, (2) the pressure of the fluid on the curved surface, and (3) the weight of the column of fluid. As in Prop. I., the pressure of the fluid on the curved surface acts perpendicular to the axis of the column, and has therefore no component along the axis.

Let p and p' denote the intensities of pressure at the points m and m' respectively. Since the section is unity, the fluid pressure on the base of the column is a force acting vertically upward equal to p; and similarly the pressure on the top of the column is a force acting vertically downward equal to p'. Now for equilibrium the upward vertical force, p, must be equal to the sum of the downward forces, p' and the weight of the column of fluid. Hence if W denotes the weight of the column of fluid,

$$p = p' + W,$$

which proves the proposition. The student will notice that this equation is true whether the fluid is of uniform or of variable density.

Cor. In the case of a weightless fluid, that is, a fluid whose weight is neglected, the pressure is the same at every point, and in all directions.

When the weight of the fluid is neglected, W is zero, and therefore $p = p'$, or the pressure is the same at all depths; and, by Prop. I., the pressure is the same at all points in the same level, so that in a weightless fluid the pressure is the same at every point. In this case the pressure at a point of the fluid is due to the pressure exerted on the fluid by rigid surfaces or the surfaces of other fluids.

We have an approximation to a weightless fluid in the case of a gas. The weight of a gas being small, we may neglect differences of pressures due to weight for small differences of level, and may consider the pressure in a small mass of gas (*e.g.* a quantity of gas in a small closed vessel) to be the same at all points.

58. Pascal's principle of the equable Transmission of fluid pressure.

The equation of the preceding Article,

$$p = p' + W,$$

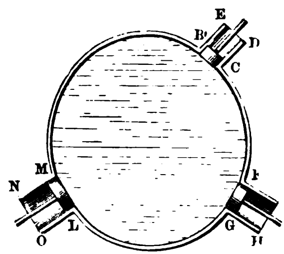
shows that if the pressure at any point of a fluid at rest is p', the pressure at any other point of the fluid differs from p' by an amount which depends only on the weight of a column of

the fluid extending between the levels of the two points. It follows that if p' is increased by any amount, p will be increased by the same amount. It follows that a pressure, exerted on a fluid at any point of the bounding surface of the fluid, is transmitted with equal intensity to all parts of the fluid. This is the characteristic property of a fluid.

This result is often referred to as Pascal's principle of the equable transmission of fluid pressure.

59. Illustrations of Pascal's Principle.

1. The figure represents a closed vessel filled with water, and BCDE, FGHK, LMNO represent cylindrical openings in the surface of the vessel, the openings being fitted with pistons BC, FG, LM respectively.



The water will exert a pressure outward on each of the pistons. Let us imagine that these pressures are balanced by the resistance of spiral springs to compression or by some other means. Then if one of the pistons, BC suppose, is pressed inward with an *additional* pressure, it is found by experiment that this additional pressure is communicated with equal intensity to the pistons FG and LM.

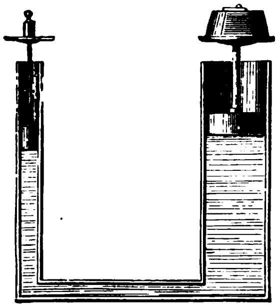
Let A_1, A_2, A_3 denote the areas of the pistons BC, GH, and LM respectively, and let P denote the *additional* pressure exerted inwards by the piston BC on the water. Then P/A_1 is the additional pressure per unit of area exerted by the piston BC on the water. If Q and R denote the *additional* outward pressures exerted by the water on the pistons FG and LM respectively, then Q/A_2 and R/A_3 are respectively the *additional* outward pressures on each unit of area of the pistons FG and LM respectively. Now it is found by experiment that

$$P/A_1 = Q/A_2 = R/A_3,$$

that is, that any pressure exerted on the water by the piston BC is communicated with equal intensity to the pistons FG and LM.

2. The figure (page 90) represents two vertical cylinders, of different sectional areas, containing water, and in communication with each other. The cylinders are fitted with air-tight pistons. If the area of the larger piston is m times that of the smaller piston, a weight of P lbs. placed on the smaller piston will support a weight of mP lbs. placed on the larger piston. The figure is drawn for the case in which the area of the larger piston is 16 times the area of the smaller, so that a weight of 1 kilogramme placed

on the smaller piston supports a weight of 16 kilogrammes placed on the larger piston. It is on this principle that we may explain the action of the **Hydrostatic Press.**



For when a weight of P lbs. is placed on the smaller piston, that piston exerts on the water a pressure of P/a lbwt. per unit of area, where a is the area of the smaller piston. By Pascal's principle this pressure is communicated with equal intensity to every unit of area in the larger piston. Hence, since the area of the larger piston is m times that of the smaller, the water will exert an upward pressure on the larger piston equal to $ma \times (P/a)$, that is, equal to mP pounds weight. Hence it follows that a weight of P lbs., placed on the smaller piston, can support a weight of mP lbs., placed on the larger piston. If W denote this weight,

$$W = mP.$$

If d and D denote the diameters of the smaller and larger pistons respectively, then

$$m = \text{area of larger piston} / \text{area of smaller piston},$$
$$= D^2/d^2.$$

Hence
$$W/P = D^2/d^2.$$

Thus the water forms part of a *machine* by means of which one weight may raise a larger weight, and it can be shown that in this, as in all other machines, the *principle of work* holds good. For if P falls through a height h, the volume of water which is forced out of the smaller into the larger cylinder is ah, and therefore the weight W rises through a height $ah/ma, = h/m$.

But the work done in raising the weight, W, through this height is

$$W \times (h/m), = mP \times (h/m), = Ph,$$
$$= \text{work done by } P \text{ on the water}.$$

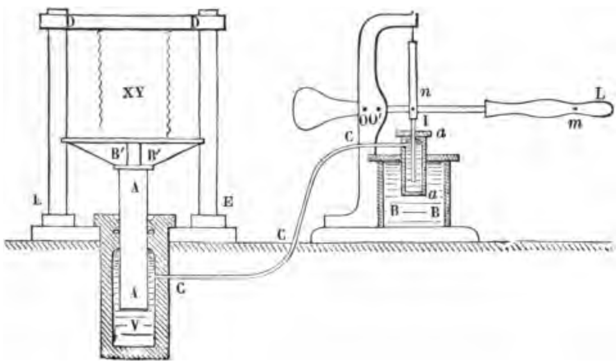
60. Bramah's Press.

The hydrostatic press in its practical form is known as Bramah's Press.

The machine consists of two parts. In one part there is a chamber BB containing water, into which is fitted a cylinder *aa*. Through the cover of this cylinder there works air-tight a rod of metal, I, called the **plunger**, which is actuated by power at the extremity *m* of a handle L, movable about the

fixed end O and attached to the plunger at *n*. At the lower end of the cylinder there is a valve opening upwards from the chamber BB, and at a point *g* in the side of the cylinder there is a valve opening outwards from the cylinder to a pipe CC communicating with the other part of the machine.

The pipe CC opens into a large cylinder V, through the top of which a solid metal cylinder AA, called the **ram**, works air-



tight. On the upper end of the ram is fixed a plate B′B′, which moves in the framework DD, EE. In using the machine as a press, the body, XY, which is to be compressed is placed on the plate B′B′, and, when the machine is worked, XY is compressed between B′B′ and the upper part DD of the framework.

The machine is worked by raising and lowering the end *m* of the lever L. When the lever is raised the plunger I is raised in the cylinder *aa*, the pressure of water on the valve at the bottom of the cylinder is reduced, and in consequence this valve opens and water passes from the chamber BB into the cylinder. On lowering the end of the lever L, the plunger I is forced down and exerts a pressure on the water in the cylinder, with the result that the valve at the bottom of the cylinder closes, the valve at the point *g* opens, and water is forced through the pipe CC into the cistern V. Thus the pump acts at first as a pump, drawing water from

the chamber BB and forcing it through the pipe CC into the cistern V, until the cistern and the connecting pipe are both full of water. On continuing to work the pump, any pressure exerted downwards on the water in the cylinder *aa* by the plunger I will be communicated with equal intensity through the water of the connecting pipe to the water in the cistern V. Thus the water in the cistern V will exert on the lower end of ram AA a pressure equal in intensity to the pressure exerted by the plunger on the water in the cylinder *aa*. The whole force exerted upwards on the ram will bear to the pressure exerted by the plunger a ratio equal to the ratio of the area of the ram to the area of the plunger. Thus the mechanical advantage of the press can be increased, theoretically at least, to any extent by increasing the ratio of the area of the section of the ram to the area of the section of the plunger.

To calculate the mechanical advantage of the press.

Let the area of the section of the ram AA be p times that of the plunger I; let Om be denoted by a, m being the point where the power is supposed to be applied, and let On be denoted by b. Thus a and b denote the distances from the fulcrum of the handle and the plunger respectively. Let a power P be exerted downwards on the lever at m, then, by the principle of the lever (Art. 37), the pressure exerted downwards by the plunger will be Pa/b.

By Pascal's principle this pressure is transmitted with equal intensity to all parts of the area of the lower end of the ram. Since the area of the section of the ram is p times that of the plunger, it follows that the ram will be pressed upwards with a force Pap/b. Hence if W denote the force of compression exerted by the press on the body XY,

$$W = Pap/b.$$

If the diameter of the plunger is d, and that of the ram D, then $p = D^2/d^2$, and therefore

$$W = PaD^2/d^2.$$

EXAMPLES.

1. What force would be exerted on the body XY in Bramah's Press if, in the notation of Art. 60, $b=1$ foot, $a=5$ feet, area of section of plunger $=\frac{1}{2}$ sq. foot, area of section of ram $=30$ sq. feet, $P=10$?

Here $$p=30/\tfrac{1}{2}=60, \text{ and } a/b=5,$$

and the mechanical advantage is therefore $5\times 60, =300$.

Hence $$W=300P, =300\times 10,$$
$$=3000 \text{ pounds weight.}$$

Hence the force exerted on the body XY will be a force of 3000 pounds weight.

2. If the ratio of the areas of the sections of the ram and plunger is 40 to 1, and if the distances from the fulcrum of the end of the lever handle and the point of attachment of the plunger are in the ratio of 3 to 1, what force will be exerted in the press when a force of 10 pounds weight is exerted on the handle?

Here $p=40$, $a/b=3$, $P=10$. Hence the force exerted in the press
$$=\ 40\times 3\times 10, =1200 \text{ pounds weight.}$$

3. The diameters of the plunger and the ram are 1 inch and 15 inches respectively, the length of the lever is 3 feet, and the distance of the point of attachment of the plunger from the fulcrum is 9 inches. What force will be exerted in the press when the lever handle is pressed down with a force of 15 pounds weight?

Ans. 15,000 pounds weight.

4. The diameters of the pistons in a Bramah's Press are 8 inches and 2 inches respectively, and the smaller piston is depressed by means of a lever 30 inches long, the distance of the point of attachment of the plunger from the fulcrum being 6 inches. If a force of 20 lbs. weight is applied vertically at the end of the lever, find the force exerted by the press.

Ans. 1600 pounds weight.

5. When the ram in Bramah's Press is exerting a very great pressure, why may a comparatively weak pipe be strong enough to contain the water which transmits the force from the plunger to the ram?

6. Deduce Pascal's principle from the application of the principle of work to the hydrostatic press. Why does not your proof apply to solids?

Pressure in Liquids, Arts. 61 to 67.

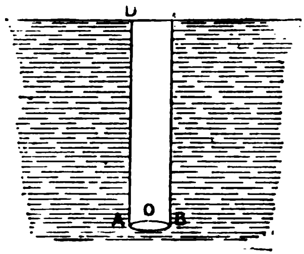
61. Variation of Pressure in Liquids at Rest.

In all questions into which the pressure of a liquid enters we assume that the liquid is incompressible and of uniform density. The following three propositions are true for liquids:—

Prop. I.—*The intensity of pressure is the same at all points of a liquid in the same level.*

This proposition has already been proved true for all fluids (Art. 57).

Prop. II.—*The difference between the intensities of pressure at two points of a liquid at different levels is proportional to the difference of levels.*



Let CD be the surface of the liquid, O be any point in the liquid, and let a unit of area AB be placed horizontally at O. Then by considering the equilibrium of the liquid column ABCD, standing on the base AB and extending to the surface CD of the liquid, we get, as in Prop. II. Article 57,

$$p = P + W,$$

where p is the pressure on the area AB, that is, the intensity of pressure at O, P the pressure downwards on the top CD of the column, and W the weight of the column. But if z represent the depth of O below the surface and w the weight of unit volume of the liquid,

$$W = wz.$$

Hence $$p = P + wz \quad\text{.................(1)}$$

Similarly if p' denote the pressure at another depth z' below CD,

$$p' = P + wz' \quad\text{.................(2)}$$

Hence, by subtracting equation (2) from equation (1), we obtain

$$p - p' = w\,(z - z') \quad\text{..............(3)}$$

which shows that the difference of pressure at two points in a liquid is proportional to the difference of depths.

Formula (1) shows that the pressure at a depth z of a liquid is the sum of wz and P, where w is the weight of unit volume of the liquid. Of these two terms, wz represents the pressure due to weight, and therefore that part of the pressure at a

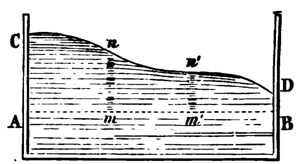
point of a liquid which is due to the weight of the liquid is proportional to the depth below the surface. The other term P represents the pressure on the surface.

The pressure P, which acts downwards on the top CD of the column ABCD of liquid, may be the pressure exerted on the surface of the liquid by a gas, or the reaction of a rigid horizontal surface against which the liquid exerts an upward pressure, or the pressure exerted on the surface by a superincumbent liquid. As examples of these three cases respectively we may take (1) the pressure of the atmosphere on the surface of water in an open vessel, (2) the pressure of a heavy piston on the surface of water in a cylinder, and (3) the pressure exerted on the surface of mercury in a vessel by a mass of water which rests on the mercury.

Prop. III.—*The free surface of a liquid at rest is horizontal.*

We assume that the atmospheric pressure is the same on all parts of the surface. The variation of atmospheric pressure over a small area, such as the surface of water in a vessel, is insensible, and may be left out of account.

Let ABDC represent a liquid; then shall the free surface CD be horizontal. Let m and m' be any two points in the liquid at the same level, and let mn and $m'n'$ be their depths respectively below the free surface. The proposition will be proved if it is proved that mn is equal to $m'n'$.



Let P denote the atmospheric pressure on the liquid, and let w denote the weight of unit volume of the liquid. Then, since m and m' are points at the same level, the pressure at m is equal to the pressure at m'. Hence

$$P + w \,.\, mn = P + w \,.\, m'n',$$

from which we get $$mn = m'n'.$$

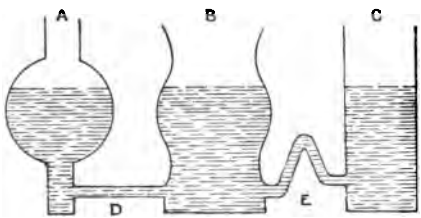
Hence any two points of the surface are at the same heights above a given horizontal plane, and therefore all points of the surface must lie in a horizontal plane.

The propositions of this Article have been proved on the assumption that vertical lines at all points of a mass of liquid are parallel. The error involved in this assumption is inappreciable in the case of a small mass of liquid, but cannot be neglected in the case of large masses of water, such

as seas and oceans. The surface of liquid at rest is really part of a surface which is nearly spherical, and although there is no sensible error in assuming that the surface of a small lake is a horizontal plane, the fact that the surface of the ocean is spherical has to be taken into account in navigation.

62. Water maintains its level.

The proposition that the surface of water at rest is horizontal is often stated under the form that "water maintains its level."



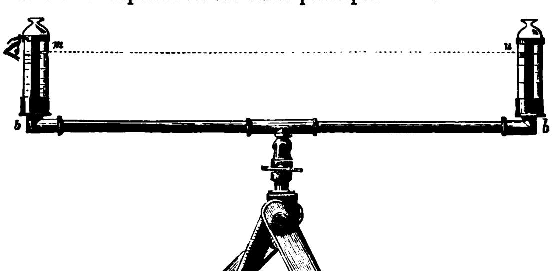
By this is meant the proposition that if there are several vessels in communication containing water, the surfaces of the water in the vessels will stand at the same level.

Thus if there are three vessels, A, B, and C, in communication, and if water be poured into the vessel A, the water will rise to the same level in the three vessels. This is true whatever be the forms of the pipes connecting the vessels.

It is on this principle that water from a reservoir is distributed throughout a district by means of pipes. It is not necessary that the pipes be laid horizontally. The highest level, however, of the pipes must be lower than the level of the water in the reservoir.

63. The water level.

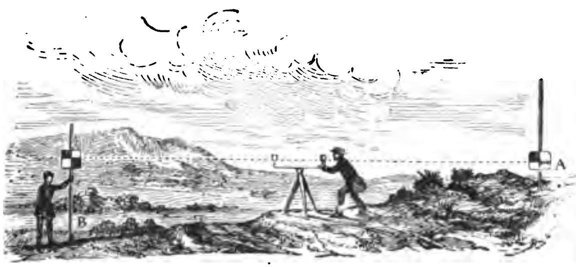
The water level depends on the same principle. In this instrument two



tubes, *bm* and *bn*, open to the air and connected by the tube *bb*, contain

water. The line, *mn*, which joins the points at which the water stands in the tubes *bm* and *bn*, is a level or horizontal line.

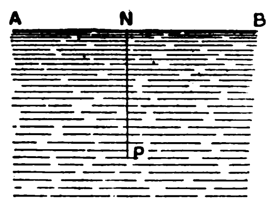
The instrument is used for determining the difference of level of two points, A and B, on sloping ground. The method of using the instrument



for this purpose is illustrated in the accompanying figure. The dotted line corresponds to the line *mn* of the preceding figure, and is horizontal.

64. Pressure at any point of a liquid whose surface is exposed to atmospheric pressure.

The atmosphere exerts a pressure on every element of the surface of a body which is freely exposed to its action. The intensity of this pressure varies from time to time and from place to place. Its average value at places not much above sea-level is about 15 lbwt. per square inch.



Let p' denote the atmospheric pressure on the surface of a liquid, w the weight of unit volume of the liquid, and z the depth PN of a point P below the surface ANB of the liquid, and p the intensity of pressure at P.

Then, by Art. 61,

$$p = p' + wz.$$

If, as is often done in Hydrostatics, we leave the atmospheric pressure out of account, the pressure at a point will be due to the weight of the liquid only, and will be given by the formula,

$$p = wz.$$

In this book, unless the contrary is explicitly stated, the

atmospheric pressure will be left out of account in the calculation of pressure at a point of a liquid.

Ex. 1.—Find the pressure in pounds weight per square inch at a depth of a mile below the surface of the sea, taking the weight of a cubic foot of water to be 62·5 lbs., and the specific gravity of sea-water to be 1·026.

Taking the inch as the unit of length, w is the weight of a cubic inch of sea-water, and z is the depth in inches. Hence

$$w = 62{\cdot}5 \times 1{\cdot}026/1728, \ z = 1760 \times 3 \times 12,$$

and the pressure

$$= 62{\cdot}5 \times 1{\cdot}026 \times 1760 \times 3 \times 12/1728,$$
$$= 2351{\cdot}25 \text{ pounds weight per square inch.}$$

Ex. 2.—What is the intensity of pressure in lbs. per square foot at a point 10 feet below the surface of water, taking into account the atmospheric pressure of 15 lbs. per square inch?

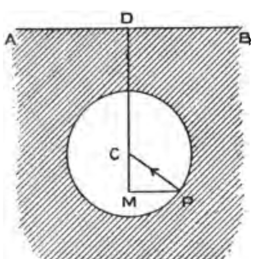
Taking a foot as the unit of length, w is the weight of a cubic foot, P is the pressure of the atmosphere on a square foot of the surface, and z is 10. Hence

$$w = 62{\cdot}5, \ P = 144 \times 15, \ z = 10,$$

and the pressure is

$$= 144 \times 15 + 62{\cdot}5 \times 10,$$
$$= 2785 \text{ lbwt. per square foot.}$$

Ex. 3.—A sphere, whose radius is 3 feet, is immersed in mercury, the depth of the centre of the sphere below the surface of the mercury being 6 feet. A small area of 1/10 of a square inch is marked on the sphere at a depth of 8 feet below the surface of the mercury. Find the vertical and horizontal components of the fluid pressure on this area.



Take a cubic inch of water to weigh 250 grains and the specific gravity of mercury to be 13·6.

Let P be the position of the small area, and let the figure represent a vertical section through P and through C, the centre of the sphere, AB being the surface of the mercury. Draw DCM, the vertical through C, and let PM be the perpendicular from P upon this line. Then

$$CP = 3 \text{ feet, } CD = 6 \text{ feet, } DM = 8 \text{ feet, and } CM = 2 \text{ feet.}$$

The pressure of the mercury is at every point perpendicular to the surface of the sphere, and therefore at a point P acts along the radius towards the centre—that is, from P to C. Let w denote the weight of a cubic inch of mercury, then

intensity of pressure at $P = w \,.\, DM$,

$$= 250 \times 13{\cdot}6 \times 8 \times 12 \text{ grains per square inch.}$$

Hence the pressure on 1/10 of a square inch at P

$$=\tfrac{1}{10} \times 250 \times 13{\cdot}6 \times 8 \times 12 = 32640 \text{ grains}.$$

The horizontal component of this pressure $= 32640 \cos \text{CPM}$ (Art. 30),

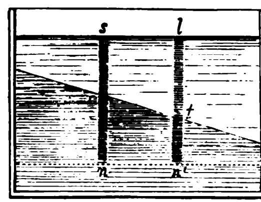
$$= 32640 . \text{PM/CP}, = 32640 . \sqrt{5}/3, = 24328 \text{ grains},$$

and the vertical component $= 32640 . \cos \text{PCM} = 32640 . 2/3 = 21760$ grains.

65. The common surface of two liquids in contact is horizontal.

This of course can be true only when the two liquids do not mix, so that there is a definite surface which is common to the liquids.

Let sl be the surface of the upper liquid; then by Prop. III. of Art. 61, sl is horizontal. Let rt be the common surface of the two liquids; it is required to prove that rt will also be horizontal.



Let n and n' be two points below the common surface, and at the same level; then the pressures at n and n' are equal. Let nrs and $n'tl$ be the vertical lines at n and n', meeting the common surface in r and t respectively. Let w and w' denote the weights of unit volumes of the lower and upper liquids respectively. Then the intensity of pressure at n is

$$w . nr + w' . rs,$$

and at n' is

$$w . n't + w' . t'l;$$

and these are equal. Hence

$$w . nr + w' . rs = w . n't + w' . tl \text{ ......(1)}$$

Also, since nn' and sl are both horizontal, $ns = n'l$, or

$$nr + rs = n't + tl \text{..............(2)}$$

Multiply equation (2) by w, and from the resulting equation subtract equation (1). We thus get

$$(w - w') . rs = (w - w') . tl,$$

showing that $$rs = tl.$$

Hence r and t, any two points of the common surface, are at

equal depths below the free surface, and therefore the common surface is horizontal.

This proposition may easily be extended to the case in which three or more liquids, which do not mix, are contained in a vessel. The common surface of any two of the liquids which are in contact will be horizontal. This is illustrated in the accompanying figure, which shows a vessel containing mercury, water, and oil, the liquids being in this order, and the mercury being lowest. The common surface of the water and the mercury, and the common surface of the water and the oil, are both horizontal.

66. Method of expressing pressure in terms of a head of a liquid.

We have seen (Art. 61) that the pressure due to weight at a point in a liquid is directly proportional to the depth of the point below the surface. On this is based a method of expressing intensity of pressure in terms of the weight of a column of a liquid. For example, the pressure of 625 lbs. per square foot is the pressure, due to weight, at a point in water at a depth of 10 feet below the surface; and we may therefore speak of this pressure as the pressure due to a **head** of 10 feet of water, or briefly as the pressure of 10 feet of water.

This method of expressing pressure is one which is in common use in stating the pressure of the atmosphere. Thus we may say that the pressure of the atmosphere at a given place and a given time is equal to the pressure of 30 inches of mercury

To compare the pressures due to heights h_1 and h_2 of two liquids whose specific gravities are s_1 and s_2 respectively.

Let w denote the weight of unit volume of water; then the pressure due to a height h_1 of a liquid of specific gravity s_1 is $h_1 \times ws_1$, and similarly the pressure due to a height h_2 of a liquid

whose specific gravity is s_2 is $h_2 \times ws_2$. Hence the ratio of the pressures is

$$h_1ws_1 : h_2ws_2, = h_1s_1 : h_2s_2.$$

Hence if h denote the height of a column of liquid of specific gravity s, the height of the column of water which would produce the *same* pressure as the column of liquid is hs.

EXAMPLE.—Into a vessel containing mercury water is poured until there is a depth of 10 inches of water. Find the pressure at a point 3 inches below the surface of the mercury, taking the specific gravity of the mercury to be 13·6.

The pressure at the point is

$=$ pressure of 10 inches of water + pressure of 3 inches of mercury,

$=$ pressure of 10 in. of water + pressure of $3 \times 13·6$ in. of water,

$=$ pressure of $(10 + 40·8)$ in. of water,

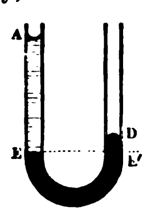
$=$ pressure of 50·8 in. of water.

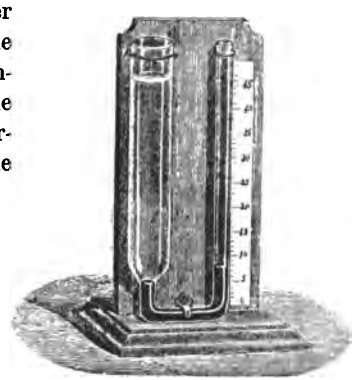
We may leave the result in this form, or we may now express this pressure in lbs. per sq. inch. Taking a cubic foot of water to weigh 62·5 lbs., we find that the pressure is 2·205 lbs. per sq. inch.

67. Equilibrium of two liquids in a bent tube.

When two liquids which do not mix are contained in a bent tube, the free surfaces of the two liquids will not stand at the same level unless the densities of the two liquids are equal.

The left-hand figure shows a bent tube, with vertical branches, containing water and mercury. AE is the part of the tube which contains water, and EE′D the part which contains mercury, the free surface of the





water being at A, and that of the mercury at D, while the surface of contact of the water and the mercury is at E.

The surface, A, of the less dense liquid, the water, stands at a higher level than the surface, D, of the mercury.

For any two liquids in a bent tube, the following proposition is true:—

Prop. *The heights of the free surfaces of the two liquids above the common surface are inversely proportional to their specific gravities.*

Taking the same figure to represent the general case, let s be the specific gravity of the less dense liquid AE, and s' that of the other EE′D. Then the pressure at E is due to a height of $AE \times s$ of water, and the pressure at E′, the point in the other branch of the tube at the same level as E, is due to a height of $DE' \times s'$ of water. But E and E′ being at the same level *in the same liquid*, the pressures at these two points are equal. Hence it follows that

$$AE \times s = DE' \times s',$$

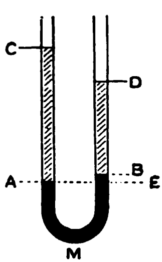
or

$$AE : DE' = s' : s,$$

which proves the proposition.

For water and mercury the ratio of s' to s is 13·6, and therefore AE is 13·6 times DE′.

The student will notice that this result does not depend on the diameter of the tube, and will still be true when the diameters of the two branches of the tube are not equal. The right-hand figure on page 101 shows the position of equilibrium of water and mercury in a bent tube, the diameters of whose branches are not equal.



Ex. 1.—The area of the cross section of a bent tube, with vertical arms, which contains mercury is 2 sq. centimetres. 20 cubic centimetres of water are poured on the mercury in one arm, and 15 cubic centimetres of alcohol are poured into the other arm. Determine the difference between the levels of the mercury surfaces in the two arms when the fluids have come to rest. The specific gravities of alcohol and mercury are ·8 and 13·6 respectively.

Let CAMD be the tube, AMB being the part occupied by the mercury, CA by the water, and BD by the alcohol.

Through A, the common surface of the water and the mercury, let AE be drawn horizontally. Then it is required to find the height above E of B, the common surface of the mercury and the alcohol. Let this height be denoted by x.

Since the area of the cross section is 2 sq. cm., it follows that 20 c. cm. of water will occupy a length of 10 cm. of the tube. Hence CA is 10 cm.; and similarly BD, the length of the tube occupied by the alcohol, is $7\frac{1}{2}$ cm.

Since A and E are at the same level *in the same liquid*, the pressures at A and E are equal. The pressure at A is a pressure of 10 cm. of water, and the pressure at E is the pressure due to the column BE of mercury and the column BD of alcohol. Hence, if $BE = x$,

the pressure at A = pressure of 10 cm. of water,
and the pressure at E = pressure of x cm. of mercury + pressure of $7\frac{1}{2}$ cm. of alcohol,
= pressure of $(13{\cdot}6 \times x + 7\frac{1}{2} \times {\cdot}8)$ cm. of water.

Therefore

$$13{\cdot}6 \times x + 7\tfrac{1}{2} \times {\cdot}8 = 10,$$

from which

$$x = 4/13{\cdot}6,$$
$$= 5/17 \text{ of a centimetre.}$$

Hence the difference of the levels of the mercury surfaces is 5/17 of a centimetre.

Ex. 2.—Two liquids that do not mix are contained in a bent tube; the difference of the levels of their free surfaces is p inches, and the height of the denser above their common surface is q inches. Compare their specific gravities.

The height of the free surface of the denser above the common surface being q inches, the height of the free surface of the other above their common surface

= q inches + difference of level,
= q inches + p inches = $(p+q)$ inches.

The specific gravities are inversely proportional to the heights of the free surfaces above the common surface, and therefore the ratio of the specific gravities is equal to the ratio of $(p+q)$ to q.

EXAMPLES VI.

[A cubic foot of water weighs 1000 oz. Neglect the atmospheric pressure unless it is explicitly referred to in the Example.]

(*The Answers are given on page* 334.)

1. Find the intensity of pressure in lbs. per square foot at a point below the surface of water at a depth of (1) 10 feet, (2) 35 feet, (3) half a mile.

2. What do the results in Example 1 become if the atmospheric pressure of 15 lbs. per square inch is taken into account?

3. The pressure on the surface of a liquid is 4 lbs. per square inch. At a depth of 6 feet it is 16 lbs. per square inch. Find the pressure at a depth of 40 feet.

4. A circle, whose radius is one foot, is described on a vertical wall of a reservoir; the surface of the water is $2\frac{1}{2}$ feet above the centre of the circle; find the ratio of the fluid pressure at the lowest point of the area to the fluid pressure at the highest point of the area.

5. Two small areas (A and B) are marked on a vertical wall of a reservoir; the area A is twice that of B, and A is 10 feet above B; if water is let into the reservoir, what will be the height of its surface above A, when the pressure of the water on A is one-fourth of its pressure on B?

6. Two equal small areas A and B are marked on a side of a reservoir full of water at different depths below the surface. The pressure on A is four times the pressure on B; but if the surface of the water in the reservoir were lowered a foot, the pressure on A would be five times the pressure on B. At what depth were A and B below the surface in the first instance?

7. If the diameter of the smaller piston in the hydrostatic press in its simplest form (see figure, page 90) is 3 inches and the diameter of the larger piston 20 inches, find what weight, placed on the larger piston, will be supported by a weight of 40 lbs. placed on the smaller piston.

8. If a force of 5 lbs. applied at the handle of the lever in Bramah's press gives a force of 3000 lbs. in the press, and if the diameters of the plunger and the ram are in the ratio of 1 to 20, what is the mechanical advantage of the lever?

9. A cubical vessel, one foot high, is half-full of mercury and half-full of water. Find the intensity of pressure in lbs. per square inch (i) at a point in the base of the vessel, (ii) at a point 3 inches below the surface of mercury.

[Specific gravity of mercury is 13·6.]

10. A spherical boiler, 4 ft. in height, is half-full of water and half-full of steam. What is the *difference* between the pressure at the bottom and top of the boiler?

11. Mercury and water are poured into a vessel, the depth of the water being 18 inches. Find the pressure in lbs. on an area of 1/16 of a square inch at a point three inches below the surface of the mercury.

12. Liquid of specific gravity 1·25 is poured into a cubical vessel until the depth is 8 inches, and on this liquid is poured 4 inches of another liquid of specific gravity 1·125. Find the intensity of pressure in lbs. per square inch at a point in the base of the vessel.

13. A vessel is partly filled with water, and then oil is poured in till there is a depth of 6 inches; find the pressure per square inch due to the weight of the liquids at a point 8·5 inches below the upper surface of the oil, assuming the specific gravity of the oil to be ·92 and the weight of a cubic inch of water to be 252 grains.

14. Two liquids which do not mix are poured into a bent tube, the ratio of the specific gravities being equal to the ratio of 5 to 4. If the height of the denser liquid above the common surface is 8 inches, what is the height of the less dense liquid above the same level?

15. A bent tube of uniform bore is held with its legs vertical; a liquid whose specific gravity is 1·8 is poured in till it stands at a height of 6 inches in each leg; another liquid which will not mix with the former, and whose specific gravity is 1·08, is now poured into one leg; how much must be poured in if the surface of the former liquid is depressed four inches?

16. In the bend of a U-shaped tube is a quantity of mercury (specific gravity, 13·6). Water is poured into one arm and glycerine into the other till their free surfaces are level. If the depths of water and glycerine be respectively 49 and 50 centimetres, find the specific gravity of glycerine.

17. Two vertical cylindrical vessels A and B are connected at the bottom by a very narrow tube, and stand on a horizontal table. The diameter of A is 6 inches, that of B is 4 inches. A liquid of specific gravity 1·4 fills the cylinders to a height of 6 inches above the base, and an equal volume of water is then poured carefully on the top of the liquid A. Where will the common surface of the liquids be when equilibrium has been restored?

18. A cylinder is put under water with its axis horizontal; a very small portion of the area of its curved surface is taken; find the horizontal and vertical components of the pressure of the water on that area. Find also numerical results in the following case:—The radius of the cylinder is 2 ft.; the axis is 4 ft. below the surface; find the horizontal and vertical components of the pressure on an area of 0·1 square inch, at a depth of 3 ft. below the surface.

19. A hollow sphere, whose internal diameter is 6 inches, is filled with water. Find the intensity of pressure in lbs. per square inch at a point on the surface 1 inch below the highest point of the water.

If an area of 1/16 of a sq. inch is marked enclosing the point, find the pressure in lbs. on this area, and state what are the vertical and horizontal components of this pressure.

20. A cone, whose height is 3 ft., and the diameter of whose base is 4 ft., is immersed in water with its axis vertical and its vertex upwards. If the depth of the vertex is 6 ft., find the intensity of pressure in lbs. per square inch at a point on the surface of the cone 8 ft. below the surface of the water.

If an area of 1/10 of a square inch is marked enclosing the point, find the pressure in lbs. on this area, and state what are the vertical and horizontal components of this pressure.

21. If the cone in Example 20 is placed with its axis horizontal, and at a depth of 10 feet in water, what is the intensity of pressure, in lbs. per square inch, at the middle point of the lowest generating line of the cone?

What are the horizontal and vertical components of the pressure on an area of 1/20 of a square inch, marked on the cone, enclosing this point?

CHAPTER VII.—WHOLE PRESSURE. RESULTANT PRESSURE.

Whole Pressure, Arts. 68 to 70.

68. **Distinction between Whole Pressure and Resultant Pressure of a Fluid on a Given Surface.**

When a surface of any form is exposed to the pressure of a fluid, we may imagine that the surface is divided into an infinite number of elements, each element being so small that it may be treated as an element of a plane. The pressure of the fluid on each element will be a force acting perpendicular to the element.

In considering the pressure of a fluid on a surface, the terms *whole* (or *total*) *pressure* and *resultant pressure* are used with the following meanings:—

(i) The **whole** (or **total**) **pressure** of the fluid on the surface **is the sum of the pressures on all the elements of the surface.**

(ii) The **resultant pressure** of the fluid on the surface **is the resultant of the pressures on all the elements of the surface,** this resultant being formed according to the rules of statics.

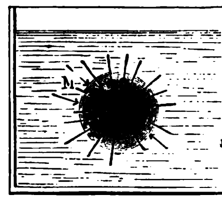
If the surface is a curved surface, the pressures on the elements of the surface will form a system of forces acting in different directions, and it is evident that the resultant of this system of forces is not equal to the sum of the forces. In this case, therefore, the whole pressure is not the same as the resultant pressure.

If the surface is a plane surface, the pressures on the elements of the surface will form a system of like parallel forces, since the force on each element acts perpendicular to the surface. Now the resultant of a system of like parallel forces is a force equal to their sum, acting in the direction of the forces. Hence we arrive at this result:—**When the surface exposed to the pressure of a fluid is a plane surface, and in that case**

only, the whole pressure coincides with the resultant pressure.

Thus in the case of a plane area *whole pressure* and *resultant pressure* mean the same thing.

As an example of a curved surface exposed to the pressure of a fluid, we may take the case of a sphere immersed in water. The water presses on each element of the surface of the sphere in a direction perpendicular to the element, that is, in the direction of the radius and towards the centre of the sphere. Hence the pressure of the water on the surface is equivalent to a system of forces acting at the centre in different directions, and it is evident that the resultant of this system of forces is not equal to the sum of the forces.



As an example of a plane surface exposed to the pressure of a fluid, we may take the side of a rectangular vessel which contains water. In this case the pressure of the water acts at every point of the side in a direction perpendicular to the side, so that the pressure of the water on the side is equivalent to a system of like parallel forces, whose resultant is equal to the sum of the forces.

69. Formula for the Whole Pressure on any Surface exposed to the Pressure of a Heavy Liquid.

Let a surface of any form be exposed to the pressure of a heavy liquid; it is required to find the **whole pressure** on the surface due to the weight of the liquid.

Let A denote the area of the surface, $\bar{z}$ the depth of the centre of gravity of the area below the free surface of the liquid, and w the weight of unit volume of the liquid.

Let the area A be divided into an infinite number of infinitesimal elements whose areas are $a_1, a_2, a_3, \ldots$ and let the depths of the centres of gravity of these elements below the surface of the liquid be $z_1, z_2, z_3, \ldots$ respectively.

The intensity of pressure on the element a_1 is wz_1, on the

element a_2 is wz_2, on the element a_3 is wz_3, and so on. Hence the pressures on the elements a_1, a_2, a_3... are respectively

$$wz_1a_1,\ wz_2a_2,\ wz_3a_3, \ldots,$$

and therefore the total pressure on the area A is

$$wz_1a_1 + wz_2a_2 + wz_3a_3 + \ldots,$$

which may be denoted in the Σ notation by $w\Sigma(za)$.

But by the formula for the co-ordinates of the centre of gravity of an area, Art. 39,

$$\bar{z} = \Sigma(za)/A,$$

and therefore

$$\Sigma(za) = A\bar{z}.$$

Hence the whole pressure on the surface

$$= w\Sigma(za), = wA\bar{z}.$$

Now $w\bar{z}$ is the intensity of pressure at the centre of gravity of the area of the surface, and therefore—

(i) *The whole pressure of a liquid on any surface is equal to the product of the number of units of area in the surface and the number which measures the intensity of pressure at the centre of gravity of the area of the surface.*

This result may be stated in another form. $A\bar{z}$ is the volume of a right prism of length $\bar{z}$, standing on a plane base of area A, and $wA\bar{z}$ is the weight of this volume of the liquid. Hence—

(ii) *The whole pressure of a liquid on any surface is equal to the weight of a right prism of the liquid, whose base is plane and equal to the area of the surface, and whose length is the depth of the centre of gravity of the surface below the surface of the liquid.*

When the surface is curved, the whole pressure is the sum of the pressures on the elements into which the surface may be divided, and does not coincide with the resultant pressure. Thus the calculation of whole pressure on curved surfaces has only a theoretical value.

When the surface is plane, the whole pressure coincides with the resultant pressure. Hence in this case $wA\bar{z}$ is the

expression for the *resultant* pressure on the area. If the surface is plane and is placed horizontally, then the pressure over the area is uniform and equal in intensity to $w\bar{z}$, so that $wA\bar{z}$ is the product of the number of units of area in the surface and the pressure on one unit of area. If the surface is plane, but not placed horizontally, the pressure will vary from point to point of the surface. In this case $w\bar{z}$ is the *average pressure*, and the resultant pressure on the surface is the product of the average pressure and the number of units of area in the surface.

Cor. If there is a pressure P on each unit of area of the surface of the liquid, this pressure will, in accordance with Pascal's principle (Art. 58), be transmitted to every unit of area of the surface A. Hence the whole pressure on the area A is

$$= wA\bar{z} + AP, = A(w\bar{z} + P).$$

In this case also $w\bar{z} + P$ is the intensity of pressure at the centre of gravity, and therefore the whole pressure is equal, as before, to the product of the area and the intensity of pressure at the centre of gravity.

In calculating whole pressure by means of the expression $wA\bar{z}$, care must be taken to express all the quantities involved in appropriate units.

If $\bar{z}$ is expressed in feet, A must be expressed in square feet, and w in units of force, say pounds weight, per cubic foot. In this case, if s is the specific gravity of the liquid, then $w = s \times 62{\cdot}5$, and the whole pressure in pounds weight will be $62{\cdot}5As\bar{z}$. If $\bar{z}$ is expressed in inches, A will be expressed in square inches, and if the weight of one pound is the unit of force, then w will be $62{\cdot}5 \times s/1728$.

If $\bar{z}$ is expressed in centimetres, then A will be in square centimetres, and if the unit of force is the weight of one gramme, then $w = s$ grammes. Hence in this case the whole pressure is $As\bar{z}$ grammes weight.

In the case of a liquid whose surface is freely exposed to the atmosphere, P will denote the intensity of atmospheric pressure.

Let h denote a height such that $P = wh$; then P is the pressure due to a

height h of the liquid. Hence a convenient method of taking the atmospheric pressure into account in the calculation of total pressure is to imagine that the atmosphere is removed, and that the depth of the liquid is increased by h, where $h=P/w$. Thus the formula for total pressure when atmospheric pressure is taken into account is

$$Aw(\bar{z}+h), \text{ where } h=P/w.$$

It is usual, however, to neglect atmospheric pressure in the calculation of the total pressure on an area pressed by a liquid. In any question, therefore, of total pressure we shall neglect the term PA unless it is expressly stated that atmospheric pressure is to be taken into account.

Ex. 1.—A cube, each of whose edges is 2 feet long, stands on one face on the bottom of a vessel containing water 4 feet deep. Find the total pressure on the upper face, and on one of the vertical faces of the cube.

The upper face is horizontal, and at a depth 2 feet below the surface. Hence taking a foot as the unit of length, we have for the upper surface

$$\bar{z}=2, A=4,$$

and w, the weight of a cubic foot of water, is 62·5 lbs. Hence the whole pressure on the upper surface

$$=w\bar{z}A=62{\cdot}5\times 2\times 4, =500 \text{ lbwt.}$$

The depth of the centre of gravity of a vertical face is 3 feet, and, therefore, for this face

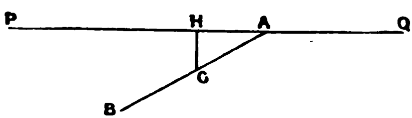
$$\bar{z}=3, A=4.$$

Hence the whole pressure on a vertical face is

$$=w\bar{z}A=62{\cdot}5\times 3\times 4, =750 \text{ lbwt.}$$

Ex. 2.—A square board, whose edge is 2 feet long, is immersed in water with one edge in the surface. If the board is inclined to the surface at an angle of 30°, find the total pressure on one side of it.

Let the figure represent the vertical section through the centre of gravity of the board perpendicular to the plane of the board, PQ representing the surface of the water, and AB the board. G, the centre of gravity of the board, is the middle point of AB, and if GH be drawn perpendicular to PQ, then GH is the depth of the centre of gravity of the board. But



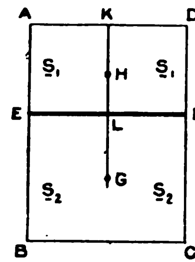
$$GH=AG \sin 30°, =\tfrac{1}{2}AG, =\tfrac{1}{4}AB,$$
$$=\tfrac{1}{2} \text{ foot.}$$

Hence the whole pressure on one side of the board

$$=\tfrac{1}{2}\times 4\times 62{\cdot}5, =125 \text{ lbwt.}$$

70. Whole pressure on the side of a vessel containing two liquids which do not mix.

Let A B C D represent the side, plane or curved, of a vessel which contains two liquids. Let E F be the common surface of the two liquids, so that E F is horizontal, and suppose that the upper liquid rises to the level A D.



Let A_1 denote the area of the upper part A E F D, and A_2 the area of the lower part E B C F, of the side. Let s_1 denote the specific gravity of the upper liquid, and s_2 the specific gravity of the lower liquid; and let w denote the weight of unit volume of water. Let H and G be the centres of gravity of the areas of the upper and lower parts, A E F D and F E B C, of the side respectively. Let H K, the depth of H below the surface of the upper liquid, be $\bar{z}_1$, and L G, the depth of G below the common surface E F, be $\bar{z}_2$. Also let K L, the depth of the upper liquid, be h. Then

whole pressure of upper liquid upon A E F D
= area of A E F D × intensity of pressure at H, (Art. 69.)

and whole pressure of lower liquid upon E B C F
= area of E B C F × intensity of pressure at G. (Art. 69, Cor.)

Now the intensity of pressure at H is the weight of a column of the upper liquid of height H K and of unit sectional area—that is, is $w s_1 \bar{z}_1$. Also the intensity of pressure at G is equal to the *sum* of the weights of columns of unit sectional area of the upper and lower liquids whose heights are K L and L G respectively. Hence the intensity of pressure at G is $(w s_1 h + w s_2 \bar{z}_2)$.

Thus the sum of the pressures exerted on the side of the vessel by the upper and lower liquid is

$$A_1 w s_1 \bar{z}_1 + A_2 (w s_1 h + w s_2 \bar{z}_2),$$
$$= w (A_1 s_1 \bar{z}_1 + A_2 s_1 h + A_2 s_2 \bar{z}_2),$$

the required expression.

This investigation holds good whether the side of the vessel is curved or plane.

If the side is plane, then the whole pressure coincides with the resultant pressure. In the case, therefore, of a plane area the above expression gives the resultant pressure of the liquids on the side of the vessel.

Ex. 1.—A cubical vessel, one metre high, is filled with mercury to a depth of 50 centimetres, and is then filled up with water. Find the pressure (i) on the base of the vessel, (ii) on a side of the vessel, taking the specific gravity of mercury to be 13·6.

Take the centimetre as the unit of length, and the weight of one gramme as the unit of force.

(i) To calculate the pressure on the base.

The intensity of pressure at any point of the base

$$= \text{pressure of 50 cm. of mercury}$$
$$+ \text{pressure of 50 cm. of water}$$
$$= 13{\cdot}6 \times 50 + 50, = 730 \text{ grammes weight per sq. cm.}$$

Hence the total (or resultant) pressure on the base

$$= 730 \times 100^2 = 7300000 \text{ grammes weight.}$$

(ii) To calculate the pressure on a side.

The area of the part of a side exposed to the pressure of the water is 50×100 sq. cm., and the depth of its centre of gravity is 25 cm. The intensity of pressure at the centre of gravity is therefore the pressure due to 25 cm. of water, that is, is a pressure of 25 grammes weight per sq. cm. Hence the whole pressure on the part of the side exposed to the water is

$$50 \times 100 \times 25, = 125000 \text{ grammes weight.}$$

The area of the part of the side exposed to the pressure of mercury is also 50×100, and the intensity of the pressure at the centre of gravity of this area is the sum of the pressure due to 50 cm. of water and the pressure due to 25 cm. of mercury, that is, is equal to $(50 + 25 \times 13{\cdot}6)$ or 390 grammes weight per sq. cm. Hence the total pressure on the lower half of a side is

$$390 \times 50 \times 100, = 1950000 \text{ grammes weight.}$$

Hence, finally, the total (or resultant) pressure on the side of the vessel is

$$= 125000 + 1950000, = 2075000 \text{ grammes weight.}$$

Ex. 2.—A cubical vessel is filled with two liquids whose specific gravities are 1 and ·8, the lower being, of course, the denser; they do not mix and their volumes are equal. Find the ratio of the resultant pressure on the upper to that on the lower half of one of the vertical faces of the cube.

Let a represent the height of the vessel, and w the weight of unit volume of water. Then the depth of each liquid is $a/2$.

Pressure on upper half of a face
$=$ area $\times$ intensity of pressure at centre of gravity of upper half of face
$= a \times a/2 \times w \times \cdot 8 \times a/4, = w a^3/10.$

Pressure on lower half of a face
$=$ area $\times$ intensity of pressure at centre of gravity of lower half of face
$= a \times a/2 \times w\ (\cdot 8 \times a/2 + a/4) = 13 w\, a^3/40.$

Hence the ratio of the pressure on the upper half is to the pressure on the lower half

$$= w a^3/10 : 13 w a^3/40, = 4 : 13.$$

Ex. 3.—A hollow sphere is full of liquid. Show that the *total pressure* on the surface of the sphere is three times the weight of the liquid.

Let w denote the weight of unit volume of the liquid, and r the radius of the sphere.

The average intensity of pressure on the surface of the sphere is equal to the intensity of pressure at the centre of gravity of the surface (Art. 69), and the centre of gravity of the surface is the centre of the sphere. The pressure at this point is the pressure due to the depth r of the liquid, that is, is a pressure of intensity $r w$. Hence the total pressure on the surface of the sphere

$=$ area of surface $\times$ intensity of pressure at centre
$= 4 \pi r^2 \times w r, = 4 \pi w r^3$......(1). (Art. 14.)

But the weight of the liquid in the sphere

$= \frac{4}{3} \pi w r^3$ ....(2). (Art. 14.)

By comparing the expressions (1) and (2) we see that the total pressure of the liquid on the surface of the sphere is three times the weight of the liquid.

EXAMPLES VII.

[A cubic foot of water weighs 1000 oz.]

(*The Answers are given on page* 334.)

1. If the whole pressure over a surface whose area is 10 sq. feet is 20880 lbs., find the measure of the average pressure when the unit of length is 2 inches.

2. Find the whole pressure on each side of a rectangular board 14 feet long and 6 feet broad immersed in water with its length horizontal, and with its upper edge 5 feet and its lower edge 9 feet below the surface of the water.

3. The water in a canal lock, 12 feet wide, and 50 feet long, is 8 feet deep. Find the pressure on the gate and on the bottom of the lock.

4. Find the total pressures in lbwt. on the bottom and on one of the sides of a cubical vessel, 3 feet deep, and filled with water.

5. What do the results become in Example 4 if the atmospheric pressure of 15 lbs. per square inch is taken into account?

6. A triangular area is immersed in a liquid with the base horizontal. The vertex is at a depth of 12 feet and its base of 9 feet. The area of the triangle is 6 square feet. If a cubic foot of the liquid weighs 70 lbs., find the total pressure on the triangle in lbwt.

7. Find in grammes weight the whole pressure on a rectangular area 8 centimetres long and 5 centimetres broad, its highest and lowest corners being at depths of 4 centimetres and 6 centimetres respectively in mercury of sp. gr. 13·6.

8. A tank, in the form of a cubical box, would hold 4 cwts. of water when quite full. Find the pressure on the base and on one of the sides of the tank when it is quite full of mercury of sp. gr. 13·6.

9. A cylinder, whose base is a circle of 6 inches radius, and whose height is 14 feet, is filled with mercury of sp. gr. 13·6. Find in lbwt. the total pressure (i) on the base, (ii) on the curved surface.

10. A vessel, of which the base is a horizontal right-angled triangle such that the sides which contain the right angle are 6 centimetres and 9·1 centimetres long respectively, and whose sides are rectangles 1·5 centimetres long, is filled with alcohol of sp. gr. ·8. Find in grammes weight the pressures on the base, and on each of the sides.

11. A rectangular vessel 12 inches deep, 10 inches long, and 7 inches wide, is half-filled with water, and is then filled to the top with oil of specific gravity ·864. Find the total pressure in pounds weight upon the bottom, and upon each of the sides, the weight of a cubic inch of water being 250 grains.

12. Into a cubical box, one foot high, water is poured until there is a depth of 4 inches, and the vessel is then filled up with oil of specific gravity ·9. Find in lbwt. the total pressure on the base, and on one of the sides of the vessel.

13. Liquid of specific gravity 1·25 is poured into a cubical vessel, one foot high, until the depth is 8 inches, and on this liquid is poured 4 inches of another liquid of specific gravity 1·125. Find in lbwt. the total pressure (i) on the base, (ii) on one of the sides of the vessel.

14. A vessel with vertical sides has a horizontal base in the form of a regular hexagon. If each side of this hexagon is 1 inch, show to what height the vessel must be filled with a liquid in order that the total pressure on each vertical side may be equal to that on the base.

15. Find the total pressure in grammes weight on a triangle of area 10 square centimetres, whose corners are at depths 3, 4, and 5 centimetres respectively in water.

16. A hollow cylinder full of water, and with both ends closed, stands with its axis first vertical, and then horizontal. Show that the total

pressure on the side of the cylinder in the first position is to the total pressure in the second position as the height is to the diameter of the base.

Also show that the total pressure on the base in the first position is to the total pressure on either end in the second position as the height is to the radius of the base.

17. A closed hollow cylinder, full of liquid, is placed with its axis (i) vertical, (ii) horizontal. Compare in each of these cases the total pressure on the curved surface with the total pressure on the base.

18. Show that if the axis is vertical, and vertex upwards, the pressure on the base of a cone, full of liquid, is equal to three times the weight of liquid.

Resultant Pressure on Plane Surfaces, Arts. 71 to 75.

71. Resultant Pressure of a Fluid on a Plane Surface. Centre of Pressure.

When a surface is exposed to the pressure of a fluid, the resultant pressure is the force which is the resultant of the pressures on the elements into which the surface may be divided. We have seen (Art. 68) that in the case of a plane surface, the **magnitude** of the resultant pressure is equal to the total pressure. Thus the resultant pressure of a heavy liquid on a plane surface is equal to $wA\bar{z}$, where A is the area of the surface, $\bar{z}$ the depth of the centre of gravity of the surface below the surface of the liquid, and w the weight of unit volume of the liquid.

The **direction** of the resultant pressure on a plane surface is perpendicular to the surface.

To complete the determination of the resultant pressure of a fluid on a plane surface, we must find its **point of application,** that is, the point of the surface at which it acts. This point is called the **Centre of Pressure.**

Def. *The centre of pressure of a plane surface exposed to the pressure of a fluid is that point of the surface at which the resultant pressure of the fluid on the surface acts.*

In the following Articles (Arts. 72 to 75) we consider the problem of finding the centre of pressure of areas immersed in heavy liquids. We shall assume in each case that there is no pressure on the surface of the liquid.

There is one case in which the position of the centre of pressure is evident, viz., the case of a plane area exposed to a

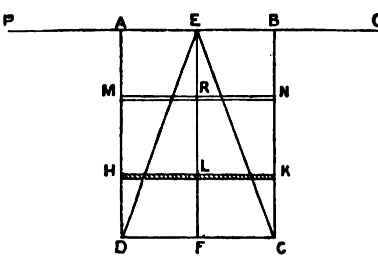
uniform pressure, *e.g.* a plane area placed horizontally in a liquid. In this case the pressure on each element is proportional to the area of the element, and therefore the resultant pressure must act at the centre of gravity of the area.

72. **Centre of Pressure of a Rectangle immersed in a Liquid with One Side in the Surface of the Liquid.**

Let ABCD be a rectangle immersed vertically in a liquid, one edge AB of the rectangle being in the surface PQ of the liquid. It is required to find the centre of pressure.

Let the area of the rectangle be divided by lines parallel to AB into an infinite number of elements in the form of very narrow strips of equal areas; and let HK and MN represent two such strips. Let EF be the straight line joining the middle points of the opposite sides AB and CD of the rectangle, meeting HK and MN in L and R respectively.

Let a denote the area of a strip, and w the weight of unit volume of the liquid. Then the pressures on the elements HK and MN are forces whose magnitudes are $wa \,.\, EL$ and $wa \,.\, ER$ respectively, whose points of application are L and R, the middle points respectively of the strips HK and MN, and whose lines of action are parallel, both being perpendicular to the plane of the rectangle. It follows that the pressures on the elements HK and MN are parallel forces proportional to the depths EL and ER respectively.



Hence the resultant pressure on the rectangle is the resultant of a system of like parallel forces, acting each at the middle point of a strip of the rectangle, the magnitude of any force of the system being proportional to the depth of the strip below E.

Now the areas of the parts of the strips HK and MN cut off by the lines EC and ED are, by similar triangles, proportional

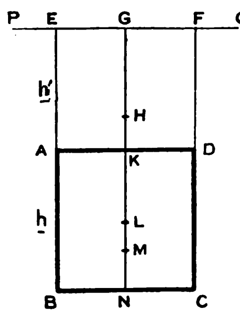
to EL and ER. Thus the areas of the strips of the *triangle ECD* at the depths EL and ER are proportional to the pressures on the corresponding elements HK and MN of the rectangle.

Thus the problem of finding the point of application of the resultant pressure on the rectangle is reduced to that of finding at what point in EF the centre of gravity of the triangle ECD lies. But it is known (Art. 40) that its distance from E is 2/3 of EF. We infer that the centre of pressure of the rectangle ABCD lies in EF at a distance from E equal to 2/3 of EF.

This result will still be true if the plane of the rectangle be rotated about the edge AB into any position inclined to the vertical. For the pressures on the elements HK and MN will at all inclinations of the plane be proportional to EL and ER.

Ex.—A rectangle ABCD is immersed vertically in a liquid, with two sides, AD and BC, horizontal. The height, AB, of the rectangle is h, and the depth of the upper horizontal side, AD, below PQ, the surface of the liquid, is h'. Find the centre of pressure of the rectangle.

Produce BA and CD to meet PQ in E and F respectively. Then $AB = h$, and $AE = h'$. Let G be the middle point of EF, and N the middle point of BC. Then GN divides the *three* rectangles EADF, ABCD, and EBCF symmetrically. Hence the centres of pressures of these three rectangles all lie in the line GN.



Let H, M, L be the centres of pressures of the rectangles EADF, ABCD, EBCF respectively. Then since the rectangles EADF and EBCF have each a side, viz. EF, in the surface of the liquid, the positions of H and L are known. By this Article

$$GH = \tfrac{2}{3}GK, = \tfrac{2}{3}h'; \quad GL = \tfrac{2}{3}GN, = \tfrac{2}{3}(h+h'),$$

and it is required to find GM. Denoting GM by H, we express that the sum of the moments about G of the resultant pressures on EADF and ABCD is equal to the moment of the resultant pressure on EBCF. Hence

$$\begin{aligned} &GL \times \text{whole pressure on EBCF} \\ &\quad = GH \times \text{whole pressure on EADF} + GM \times \text{whole pressure on ABCD}. \end{aligned}$$

The whole pressures on the areas EBCF, EADF, ABCD are proportional respectively to the products obtained by multiplying each of these

areas by the depth of its centre of gravity (Art. 71). The areas are proportional to $(h+h')$, h' and h respectively; and the depths of their centres of gravity are proportional to $(h+h')$, h', $(h+2h')$ respectively. Hence the equation of moments leads to

$$\tfrac{2}{3}(h+h') \cdot (h+h')^2 = \tfrac{2}{3}h' \cdot h'^2 + H \cdot h(h+2h'),$$

from which we get

$$H = \frac{2}{3}\,\frac{(h+h')^3 - h'^3}{h(h+2h')} = \frac{2}{3}\,\frac{h^3+3h^2h'+3hh'^2}{h(h+2h')},$$
$$= 2(h^2+3hh'+3h'^2)/3(h+2h'), \qquad \text{(1)}$$

which determines GM, the depth of the centre of pressure of the area ABCD.

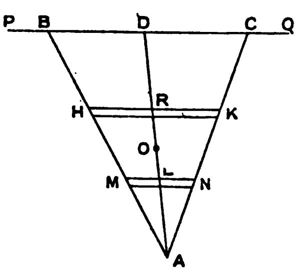
If $h'=h$, the formula for H becomes

$$H = 14\,h/9, \qquad \text{(2)}$$

which determines the depth of the centre of pressure of a rectangle immersed vertically in a liquid with two edges horizontal, the upper horizontal edge being at a depth below the surface equal to h, the height of the rectangle. The student may easily verify this result by a direct application to this case of the equation of moments.

73. Centre of Pressure of a Triangle immersed in a Liquid with One Edge in the Surface of the Liquid.

Let ABC be a triangle immersed in a liquid with one side, BC, in the surface, PQ, of the liquid. It is required to find the centre of pressure.



Join A to D, the middle point of BC, and divide the area of the triangle into narrow strips, of equal breadths, by lines parallel to BC. Let MN and HK be two strips at equal distances from A and D respectively, and let MN and HK meet AD in L and R respectively. Then AL is equal to DR, and AR is equal to DL.

The strips MN and HK may be taken to be rectangles; and therefore their areas, since they are of equal breadths, are proportional to HK and MN. But by similar triangles HK and

MN are proportional to AR and AL. Hence

$$\text{area of HK} : \text{area of MN} = \text{AR} : \text{AL} \quad \ldots\ldots\ldots(1)$$

Since D is the middle point of BC, it follows that AD bisects each of the lines drawn parallel to BC. Hence R and L are the middle points respectively of HK and MN, and the pressure on these elements may be taken to be forces acting at R and L. Let p and p' be the *intensities of pressure* at R and L respectively. Then p and p' are proportional to the depths of R and L, and therefore proportional to DR and DL. Hence

$$p : p' = \text{DR} : \text{DL} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From equations (1) and (2) we get, by compounding ratios (remembering that DR = AL and DL = AR),

$$p \times \text{area of HK} : p' \times \text{area of MN} = 1 : 1,$$

or

$$p \times \text{area of HK} = p' \times \text{area of MN}.$$

Hence the pressure on the element HK is equal to the pressure on the element MN, and the resultant of these two pressures will be a force acting at the middle point of RL. Let this point be denoted by O, then O is also the middle point of AD, for AL = DR.

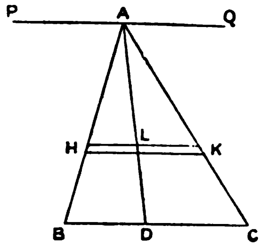
In this way by taking pairs of strips at equal distances from A and D respectively, we see that the resultant pressure on the triangle will be the resultant of a system of forces acting at O, that is, the resultant pressure on the triangle is a force acting at O. Hence the centre of pressure of the triangle is O, the middle point of the median AD of the triangle.

This investigation evidently holds good whatever be the inclination of the plane of the triangle to the vertical, for at all inclinations the pressures at R and L will be proportional to DR and DL.

74. Centre of Pressure of a Triangle immersed in a Liquid with Vertex in the Surface and Base Horizontal.

Let ABC be a triangle immersed in a liquid with the vertex

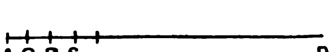
A in the surface of the liquid, and the base BC horizontal. It is required to find the centre of pressure.



As in the preceding Article, let the area of the triangle be divided, by lines parallel to BC, into an infinite number of narrow strips of equal breadths, and let HK be any one of these strips. Take D, the middle point of BC, and draw the median AD, cutting HK in L. Then AD bisects the areas of all the strips, and therefore the centre of pressure must lie in AD.

The area of HK is proportional to AL, and the intensity of pressure at L is proportional to AL. Hence the pressure on the strip HK is proportional to AL^2. It follows that the centre of pressure of the triangle will coincide with the centre of gravity of an infinite number of particles, arranged at equal distances along AD, the weight of each particle being proportional to the square of its distance from A. We proceed to find the centre of gravity of this system of particles by the method of integration (Chap. V.).

Let AD be divided into n equal parts at the points Q, R, S,... Let $AD = c$, and let $c = nh$, so that h is the length of each of the parts AQ, QR, RS,... Let masses be placed at Q, R, S,... proportional to the squares on AQ, AR, AS,... that is, proportional to $1^2, 2^2, 3^2, \ldots$ respectively. The distances of these masses from A are h, $2h$, $3h$,... respectively. Let $\bar{x}$ denote the distance of their centre of gravity from A, then by the formula



$$\bar{x} = \Sigma(mx)/\Sigma m, \ldots \text{(Art. 39)},$$

we get

$$\bar{x} = (h \cdot 1^2 + 2h \cdot 2^2 + 3h \cdot 3^2 + \ldots)/(1^2 + 2^2 + 3^2 + \ldots),$$

$$= h\,\Sigma n^3/\Sigma n^2,$$

$$= h\frac{n^2(n+1)^2}{4} \Big/ \frac{n(n+1)(2n+1)}{6} \qquad \text{(Art. 42)}$$

$$=\frac{3h}{2}\cdot\frac{n(n+1)}{2n+1}, \quad =\frac{3nh}{2}\cdot\frac{1+1/n}{2+1/n}.$$

Now $nh=c$, and on making n infinite, $1/n$ becomes zero. Hence

$$\bar{x}=\frac{3c}{2}\cdot\frac{1}{2}=\frac{3}{4}c.$$

Hence we have arrived at the result that the centre of pressure lies in A D, at a distance from A equal to 3/4 of A D.

75. Formula for the Depth of the Centre of Pressure of any Plane Area.

Let the area be divided, as in Art. 69, into elements. Then, using the notation of that Article, the centre of pressure will coincide with the centre of gravity of a system of particles, whose weights are proportional to $a_1 z_1, a_2 z_2, a_3 z_3, \ldots$ and whose depths below the surface of the liquid are $z_1, z_2, z_3, \ldots$ respectively. Hence, if H denote the depth of the centre of pressure, it follows from the formulæ of Art. 39 that

$$H=\Sigma\,(az^2)/\Sigma\,(az). \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Let the plane of the area, produced if necessary, cut the surface of the liquid in the line P Q. Then if the plane of the area is vertical, $z_1, z_2, z_3, \ldots$ are the lengths of the perpendiculars from the elements $a_1, a_2, a_3, \ldots$ respectively upon P Q, and the value of H, found from (1), will be the length of the perpendicular from the centre of pressure upon P Q.

If the plane of the area is inclined to the vertical, the depths $z_1, z_2, z_3, \ldots$ will be *proportional* to the perpendiculars from the elements upon P Q, and may be taken to represent these perpendiculars. The value of H, found from (1), will then represent the length of the perpendicular from the centre of pressure upon P Q.

Formula (1) may be written in another form. Let k denote the radius of gyration of the area about the line P Q, and let $\bar{z}$ denote the perpendicular from the *centre of gravity* of the area upon P Q. Then, A denoting the area, we have

$$Ak^2=\Sigma\,(az^2), \text{ and } A\bar{z}=\Sigma\,(az);$$

therefore

$$H=Ak^2/A\bar{z},$$
$$=k^2/\bar{z}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Either of the formulæ (1) and (2) will determine a horizontal line in the area in which the centre of pressure must lie. In many cases it is easy, from considerations of symmetry, to find another line in the area in which this point must lie. The point of intersection of these two lines will be the centre of pressure. Thus, for example, in the cases discussed in

Arts. 73 and 74, it is evident that the centre of pressure lies in the median A D.

We append some examples of the use of formula (2):—

(i) Rectangle with edge in the surface.

Let h represent A D, the height of the rectangle. (See figure, Art. 72.) Then

$$k^2 = h^2/3 \text{ (Art. 51), and } \bar{z} = h/2. \quad \text{(Art. 40.)}$$

Hence $$H = h^2/3 \div h/2, = 2h/3,$$

which agrees with the result in Art. 72.

(ii) Triangle with side in the surface.

Let h denote the *perpendicular* from A upon B C. (See figure, Art. 73.) Then

$$k^2 = h^2/6 \text{ (Art. 51), and } \bar{z} = h/3. \quad \text{(Art. 40.)}$$

Hence $$H = h^2/6 \div h/3, = h/2,$$

which agrees with the result in Art. 73.

(iii) Triangle with vertex in surface and base horizontal.

As in (ii), let h denote the perpendicular from A upon B C. (See figure, Art. 74.) In this case k is the radius of gyration of the triangle about P Q. The square of the radius of gyration about B C is $h^2/6$, about a parallel axis through the centre of gravity is $h^2/6 - (h/3)^2$, and therefore about P Q is $h^2/6 - (h/3)^2 + (2h/3)^2$. (Art. 50.) Hence

$$k^2 = h^2/6 - h^2/9 + 4h^2/9, = h^2/2,$$

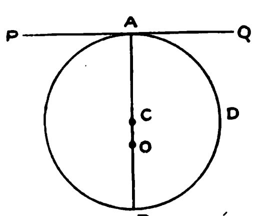
and $$\bar{z} = 2h/3. \quad \text{(Art. 40.)}$$

Hence $$H = k^2/\bar{z}, = h^2/2 \div 2h/3, = 3h/4,$$

which agrees with the result in Art. 74.

(iv) Circle just immersed.

Let A, the highest point of a circle A D B, be in the surface P Q of a liquid. Then P Q is a tangent to the circle at A. Let C be the centre, and let the radius A C be denoted by a.



It is evident from symmetry that the centre of pressure lies in the diameter A C B. Let it be O; then it is required to find H, the distance A O.

Here k^2 = square of the radius of gyration of circle about P Q,

$$= 5a^2/4, \text{ (Art. 51.)}$$

and $$\bar{z} = \text{A C}, = a.$$

Hence $$H = 5a^2/4 \div a, = 5a/4.$$

Hence the centre of pressure lies in the radius drawn to the lowest point of the circle, at a distance from the centre equal to 1/4 of the radius.

EXAMPLES VIII.

(*The Answers are given on page* 334.)

1. Show from general considerations that the centre of pressure of a plane area is always at a lower depth than the centre of gravity of the area.

2. A rectangular hole A B C D, whose lower side C D is horizontal, is made in the side of a reservoir, and is closed by a door whose plane is vertical. The door can turn freely round a hinge coinciding with C D. Calculate the force that must be applied to AB to keep the door shut, taking A B to be 1 foot and A D to be 12 feet long, and supposing that the water rises to the level of A B.

3. ABCD is a square, and O is the point of intersection of the diagonals A C and B D. The square is placed with the edge A B in the surface of a liquid, and with its plane vertical. Find in terms of a, the side of the square, the depth below the surface of the centre of pressure of—

 (i) the triangle BOC,
 (ii) the triangle COD,
 (iii) the five-sided figure ABCDO,
 (iv) the five-sided figure ADOCB.

4. If the square in the preceding Example is placed with the angular point A in the surface of the liquid, and the diagonal A C vertical, find the depth below the surface of the centre of pressure of—

 (i) the square,
 (ii) the triangle ABC,
 (iii) the triangle BOC,
 (iv) the triangle BCD,
 (v) the five-sided figure ABCDO.

5. A B C D is a rectangle immersed vertically with the side A D in the surface of a liquid; E is the middle point of A D, F and G are the middle points of the vertical sides A B and D C respectively. Find in terms of h, the height A B of the rectangle, the depth below A D of the centre of pressure of—

 (i) the triangle EFB,
 (ii) the triangle EGB,
 (iii) the triangle BGC,
 (iv) the quadrilateral EFBG,
 (v) the quadrilateral EBCG,
 (vi) the five-sided figure EFBCG.

6. A circle is immersed in a liquid with its plane vertical, and its centre at a depth below the surface equal to the length of the diameter. Find in terms of a, the radius, the depth below the surface of the centre of pressure of the circle.

Resultant Pressure on Curved Surfaces, Arts. 76 and 77.

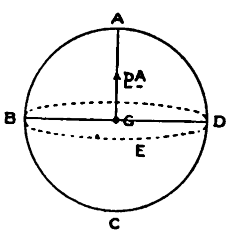
76. Resultant Pressure on a given part of a Closed Vessel filled with Weightless Fluid.

A closed vessel is filled with *weightless* fluid; it is required to determine the resultant pressure of the fluid on a part of the surface cut off by a given plane.

[We may suppose the fluid to be a gas, such as steam, since the pressure exerted by a gas on the sides of the vessel which contains it is mainly due to the tendency of the gas to expand. Thus the pressure due to weight in a small mass of gas may be neglected, and we may therefore take a gas as an example of a weightless fluid.]

Let A B C D be the vessel, and let B E D be a plane cutting off the part B A D of the surface. To find the resultant pressure of the fluid on B A D.

Consider the equilibrium of the mass of fluid in ABD. This fluid is at rest under the action of the fluid pressure across the plane B E D and the reaction of the surface B A D on the fluid. Hence these forces must be equal and opposite.



Let p denote the intensity of pressure at any point of the fluid. Then, since the fluid is without weight, the pressure at every point is p. Hence the pressure across the plane B E D is equal to $p\,A$, where A is the area of the plane, and acts at the centre of gravity of the plane B E D in a direction perpendicular to the plane (Art. 71). If G is the centre of gravity of the plane B E D, and G A is drawn perpendicular to the plane, meeting the surface B A D in A, then the pressure across the plane B E D is a force equal to $p\,A$, and acts in the direction G A. The reaction of the surface B A D on the fluid is, as we have seen, an equal and opposite force. But the pressure of the fluid on the surface B A D is equal and opposite to the reaction of the surface on the fluid. It follows that the pressure of the fluid on the surface is a force $p\,A$ acting at the point A in the direction from G to A.

Ex.—A hollow sphere is filled with fluid. Neglecting the weight of the fluid, find the resultant pressure on a given half of the surface.

Taking the above figure to represent the sphere, let G be the centre and B E D a diametral plane. Then the resultant pressure of the fluid on the hemisphere B A D is a force acting at A, the vertex of the hemisphere, along the radius G A, equal in magnitude to pA, where p is the pressure of the fluid, and A is the area of the diametral section B E D. If r is the radius of the sphere, $A = \pi r^2$, and therefore the resultant pressure is $\pi p r^2$.

As a numerical example, suppose that p is 15 lbwt. per square inch, and that r is 1 foot, $=12$ inches. Then the resultant pressure on one half of the spherical surface is

$$= 22/7 \times 15 \times 144, = 6788\tfrac{4}{7} \text{ lbwt.}$$

77. Resultant Pressure of a Heavy Liquid on a Given Curved Surface.

When a given curved surface is exposed to the pressure of a heavy liquid, the resultant pressure of the liquid on the surface may be found by finding the components of this pressure in any three directions at right angles, and compounding these components, when *concurrent*, by the parallelogram law.

It is convenient to choose as one of the directions the direction of the vertical. The other two directions must then be taken to be the directions of two horizontal lines at right angles to each other, which may be chosen arbitrarily.

The component in the vertical direction of the pressure on the surface is called the **resultant vertical pressure** of the liquid on the surface. It is found by *projecting* the given surface on the surface of the liquid.

By drawing vertical lines through every point of the boundary of the curved surface to the surface of the liquid, we shall enclose a column of liquid standing on the given surface. This column will be in equilibrium under the action of (i) its weight, (ii) the pressure of the curved surface, (iii) the pressure on its sides of the surrounding liquid. (Art. 54.) The force (iii) acts horizontally, since the sides of the column are vertical, and has therefore no vertical component. It follows that the weight of the column, which is a vertical force, is balanced by the vertical component of the pressure of the surface on the liquid. But the pressure of the surface is equal and opposite to

the pressure of the liquid on the surface. Hence the resultant vertical pressure of the liquid on the surface is equal to the weight of the liquid in the column formed by drawing vertical lines through every point of the boundary of the surface, and acts through the centre of gravity of this column.

In this statement regarding the resultant vertical pressure we have supposed that the vertical component of the pressure on each element of the surface acts downwards. In some cases this component acts upwards, and in some other cases it acts upwards on some parts of the surface and downwards on other parts. (See Ex. 2 below.)

Having shown how to find the resultant vertical pressure on the surface, it remains to consider how the resultant horizontal pressure on the surface may be found.

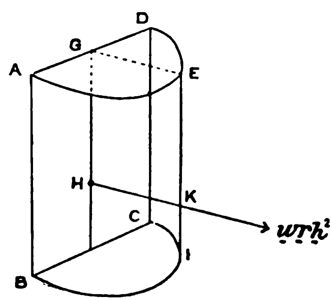
The resultant pressure of the liquid on the surface in a *given* horizontal direction is found by drawing any plane perpendicular to the given direction, and *projecting* the surface on this plane We draw through each point of the boundary of the surface a line perpendicular to the plane so as to enclose by horizontal lines a part of the liquid, bounded at one end by the curved surface and at the other by the plane. By considering the equilibrium of this liquid, we see that the resultant horizontal pressure, in the given direction, of the liquid on the surface is equal to the pressure on the end which is plane. For the weight of the liquid acts vertically, and the pressures of the surrounding liquid all act perpendicular to the given direction. Hence the resultant horizontal pressure in the given direction on the curved surface is equal to the pressure on the plane, and may therefore be found By choosing two directions at right angles, we can find in this way the components of the resultant pressure in these two directions, and, if these forces meet in a point, the **resultant horizontal pressure** of the liquid on the surface will be found by compounding them by the parallelogram law.

Finally, by compounding the resultant vertical pressure with the resultant horizontal pressure, we find **the resultant pressure of the liquid on the surface.**

In general, the determination of the resultant pressure of a liquid on a curved surface is a problem of great difficulty. In some of the simpler cases the resultant horizontal pressure is zero, and the problem is reduced to that of finding the resultant vertical pressure. In other cases, again, although the resultant horizontal pressure is not zero, it is easy to see what is the direction of this component of the resultant pressure, so that the problem is somewhat simplified.

Ex. 1.—A hollow cylinder is placed with its axis vertical, and is filled with water. To find the resultant pressure of the water on either of the two parts into which the curved surface is divided by a plane through the axis of the cylinder.

Let the figure represent one of the halves into which the cylinder is divided by a plane through the axis. ABCD represents this plane, and BFCDEA is one half of the curved surface of the cylinder. Consider the equilibrium of the water which fills this half of the cylinder. This water is in equilibrium under the action of four forces:—(i) its weight, (ii) the reaction of the base, (iii) the pressure which is exerted on the water in this half of the cylinder by the water in the other half of the cylinder, a force acting at H, the centre of pressure of the rectangle ABCD, perpendicular to the plane ABCD, (iv) the reaction of the curved surface BFCDEA.

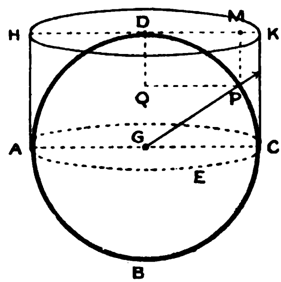


Of these four forces (i) and (ii) act vertically, (iii) and (iv) act horizontally. Hence the forces (i) and (ii) balance, and the forces (iii) and (iv) balance. Now the force (iii) is the pressure across the plane ABCD, and is a force equal to wrh^2, where r is the radius and h the height of the cylinder (Art. 69). This force acts at H, the centre of pressure of the rectangle ABC, whose depth HG is $2h/3$ (Art. 72). The reaction of the curved surface is an equal and opposite force, and the *resultant pressure of the water on the curved surface*, which is equal and opposite to the reaction of the curved surface on the water, is therefore a force wrh^2, acting along the line drawn from H perpendicular to ABCD towards the curved surface. In the figure this line is HK.

Ex. 2.—A hollow sphere is filled with liquid. Find the resultant pressures of the liquid on the two parts into which the surface is divided by the horizontal plane through the centre.

Let ABCD be the sphere, and AEC the horizontal plane through the centre G of the spherical surface. It is required to find the resultant pressure of the liquid (i) upon the upper hemisphere ADC, (ii) upon the lower hemisphere ABC.

Let P be a small element of the upper hemisphere; then the pressure of the liquid on this element is a force perpendicular to the surface at P, that is, a force in the direction GP, whose magnitude depends, for different positions of P, upon the depth DQ of P below D, the highest point of the liquid. Now imagine a cylinder ACKH erected on the base AEC, of height AH equal to the radius of the sphere, and imagine that the space between the cylinder and the sphere is filled with liquid. Then the pressure exerted by this liquid on any element of the surface of the sphere would be equal and opposite to the pressure exerted by the liquid inside the sphere on the other side of the same element. For the pressure due to the liquid outside the sphere on an element of the surface at the point P would be a pressure due to the depth PM, acting in the direction PG. It follows that the resultant pressure of the liquid inside the sphere on the upper hemisphere ADC is equal and opposite to the resultant pressure of the liquid between the cylinder and the sphere on the surface of the sphere. The *resultant* reaction of the sides of the cylinder on latter liquid is zero (apply result Ex. 1). The other forces acting on this liquid are (i) its weight, (ii) the reaction of the spherical surface ADC. Hence the reaction of the spherical surface is equal to the weight of the liquid between the sphere and the cylinder, and acts vertically upwards through the centre of gravity of this liquid. The pressure of this liquid on the sphere is an equal and opposite force, that is, is a force acting downwards. Hence, finally, the resultant pressure of the liquid inside the sphere on the hemisphere ADC is an *upward* force equal to the weight of the liquid between the cylinder and the sphere, and acts along the line GD.



Let w denote the weight of unit volume of the liquid, and r the radius of the sphere. Then if R_1 denote the resultant pressure of the liquid inside the sphere on the upper hemisphere ADC,

$$R_1 = \text{weight of liquid between the cylinder HACK and the sphere ADC,}$$
$$= (\pi r^2 . r - \tfrac{2}{3}\pi r^3)w, \qquad \text{(Art. 14)}$$
$$= \tfrac{1}{3}\pi r^3 w, \;= \tfrac{1}{4} . (\tfrac{4}{3}\pi r^3 w), \;= \tfrac{1}{4}W,$$

where W is the weight of liquid in the sphere (Art. 14). In finding the resultant pressure on the lower hemisphere ABC, R_2 say, we may imagine that the upper hemispherical surface ADC is removed, and replaced by the cylinder HACK containing liquid to the level HDK. For, as we have just seen, the surface ADC presses down on the liquid inside the sphere with a force equal to the weight of the liquid between the sphere and the cylinder, and this pressure will not be altered by replacing the rigid surface of the hemisphere by the column of liquid between the cylinder and the

sphere. It follows that the resultant pressure of the liquid inside the sphere on the lower hemispherical surface ABC is equal to the sum of the weights of the liquid in ABC and in the cylinder HACK. Hence

$$R_2 = (\tfrac{2}{3}\pi r^3 + \pi r^2 . r)w, \ = \tfrac{5}{3}\pi r^3 w, \qquad \text{(Art. 14)}$$
$$= \tfrac{5}{4}(\tfrac{4}{3}\pi r^3 w), \ = \tfrac{5}{4}W.$$

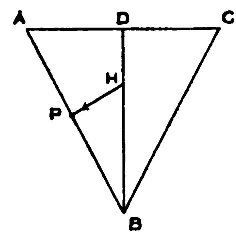
Having thus found the resultant pressures on each of the two hemispheres, we may now find the resultant pressure of the liquid inside the sphere on the surface of the sphere. This resultant is evidently the resultant of the downward pressure on the lower hemisphere and the upward pressure on the upper hemisphere. Thus the resultant pressure on the surface of the sphere

$$= R_2 - R_1, \ = \tfrac{5}{4}W - \tfrac{1}{4}W,$$
$$= W.$$

Hence the resultant pressure of the liquid on the surface of the sphere is equal to the weight of the liquid. This result may be obtained immediately by considering the equilibrium of the liquid in the sphere.

Ex. 3.—A vessel in the form of a hollow cone with its axis vertical and vertex downwards, is filled with liquid. Find the resultant pressure of the liquid on the surface of the cone.

Here the weight of the liquid is evidently equal and opposite to the reaction of the sides of the vessel on the liquid, and this reaction is equal and opposite to the pressure of the liquid on the sides. Hence the pressure of the liquid on the sides is equal to the weight of the liquid and acts vertically along the axis.



This result may also be obtained by using the formula for *whole* pressure (Art. 69). The whole pressure of the liquid on the sides of the vessel is $wA\bar{z}$, where w is the weight of unit volume of the liquid, A the area of the surface of the cone, and $\bar{z}$ the depth of the centre of gravity of this area below the surface of the liquid.

Now $\bar{z} = h/3$, where h is the length of the axis DB of the cone. [This follows immediately by considering that the surface ABC of the cone may be divided into elements in the form of triangles, all having their vertices at B.] Also $A = \pi r\sqrt{r^2 + h^2}$ (Art. 14), where r is the radius AD of the base of the cone. Hence the *whole* pressure is

$$\tfrac{1}{3}w\pi hr\sqrt{r^2 + h^2}.$$

Now the resultant pressure is the resultant of the vertical components of the pressures on the elements of the surface. If P is an element of the surface of the cone, the pressure at P acts along HP, a line drawn through P perpendicular to AB. Hence the vertical component of the pressure on the element at P will be found by multiplying the pressure on the element

at P by the cosine of the angle PHB (Art. 30), that is, by the sine of the angle ABD, that is, by $r/\sqrt{r^2+h^2}$. The same thing is true for every element of the surface, and therefore the resultant vertical pressure on the surface of the cone, which in this case is the resultant pressure, may be found by multiplying the whole pressure by $r/\sqrt{r^2+h^2}$. The result is $\frac{1}{3}w\pi r^2h$, which is equal to the weight of liquid in the cone.

Principle of Archimedes, Arts. 78 to 80.

78. Resultant Pressure on a Body wholly or partly immersed in a Fluid.

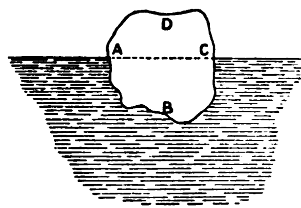
Prop. *When a body is wholly or partly immersed in a fluid, the resultant pressure of the fluid on the body is an upward force, whose magnitude is equal to the weight of the fluid displaced, and whose line of action is the vertical line through the centre of gravity of the fluid displaced.*

For if we imagine the body to be removed, and the mass of fluid which it displaces to be restored, the pressures of the surrounding fluid on the surface of this mass of fluid would be precisely the same as the fluid pressures on the surface of the body, and therefore the *resultant* pressure on the body is equal in all respects to the resultant pressure which would be exerted on the replaced fluid. But this mass of fluid would be in equilibrium under the action of (1) its weight and (2) the resultant pressure of the surrounding fluid (Art. 54). For equilibrium these two forces must be equal and opposite. Hence the resultant pressure of the surrounding fluid on the replaced fluid must be an upward force equal to the weight of this fluid, and must act in the vertical line through the centre of gravity of this fluid.

It follows that the resultant pressure of the fluid on the *body* is an upward force, whose magnitude is equal to the weight of the fluid displaced, and whose line of action is the vertical line through the centre of gravity of this fluid.

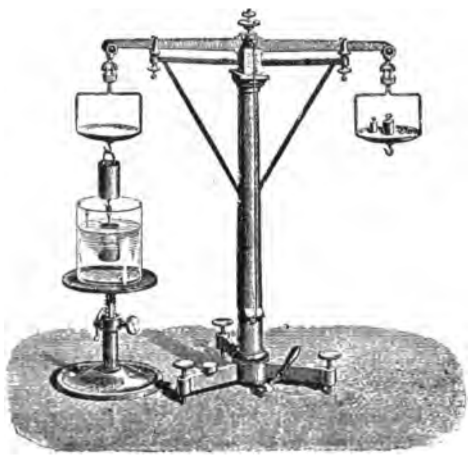
This reasoning holds good whether the fluid is a liquid or a gas, and whether the body is wholly immersed or only partly immersed. The two cases of total and partial immersion are represented by the figure on page 107 and the accompanying

figure respectively. In the latter figure, which represents a body partly immersed in a liquid, the part ABC is the part of the surface of the body which is in contact with the liquid. The pressure of the liquid on the body is equal to the weight of the liquid ABC displaced, and acts vertically upwards through the centre of gravity of this liquid.



This proposition is referred to as the **principle of Archimedes.** It gives (1) the **magnitude,** (2) the **direction,** (3) the **line of action** of the **resultant pressure of a fluid on a body wholly or partly immersed in the fluid.**

It follows that if a body is wholly immersed in a fluid, and then left free to move, it will rise or sink in the fluid according as its weight is less than or greater than the weight of fluid which it displaces, and will remain at rest if its weight is just equal to this weight of fluid.



Experimental Verification of Principle of Archimedes.

79. Experimental Verification of the Principle of Archimedes.

From one of the scale-pans of a balance there is suspended a small hollow cylinder, and from this cylinder is suspended a solid cylinder of metal, which just fits into the hollow cylinder. The volume of the solid cylinder is therefore just equal to the volume of the space inside the hollow cylinder.

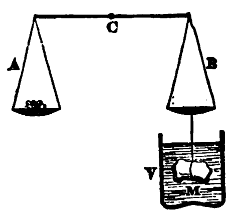
The two cylinders are counterpoised by weights placed in the other scale-pan of the balance.

The solid cylinder is now immersed in a beaker of water, and it is then found that the weights preponderate, and that the solid cylinder rises in the water. On pouring water slowly into the hollow cylinder, the solid cylinder slowly sinks again in the water, and a counterpoise is again obtained when the hollow cylinder is full of water, and the solid cylinder is wholly immersed.

This experiment clearly shows that the water presses the solid cylinder upwards with a force equal to the weight of water which fills the hollow cylinder; and this weight is equal to the weight of water which the solid cylinder displaces.

80. The Hydrostatic Balance.

If a body M would sink in a liquid, and if it be supported in the liquid by a fine thread attached to B, one of the scale-pans of a balance ACB, the weight of the mass which must be placed in the scale-pan A to counterpoise M is said to be the *apparent weight of the body M in the liquid.* It follows from the principle of Archimedes that the apparent weight of the body in the liquid is less than the true weight of the body by the weight of the liquid displaced by the body.



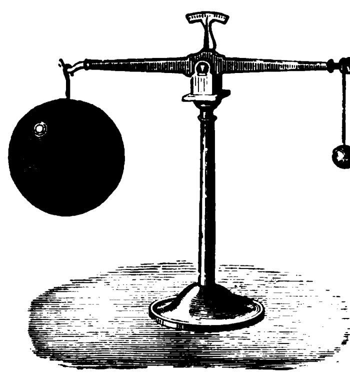
The balance, when used for finding the apparent weight of a body in a liquid, is called the *hydrostatic balance.* Some of the methods by which the specific gravities of bodies are experimentally determined require the use of the hydrostatic balance.

Since, by the principle of Archimedes, a body in air is pressed upwards by the air with a force equal to the weight of air which the body displaces, it follows that the result obtained on weighing a body in air is not the true weight of the body. For both the body and the mass which counterpoises it are pressed upwards by the atmosphere, and if their volumes are not equal, the upward forces exerted on them by the air are not equal. In order to find from the weight of a body in air the true weight, that is, the weight *in vacuo,* we must make corrections for the weights of air displaced by the body and the mass which counterpoises it. These corrections are very small,

and will, unless the contrary is expressly stated, be neglected in this book. We shall therefore assume that the weight of a body in air is the true weight of the body.

The following experiment illustrates the upward pressure of the atmosphere. From one end of the beam of a form of the balance, known as the *baroscope*, is suspended a small brass ball, and from the other end a larger ball of cork. The two balls exactly counterpoise each other in air, but when the balance is placed under the receiver of an air-pump, and the air withdrawn from the receiver, the balls no longer counterpoise each other, the ball of cork sinking and the other ball rising. When the air is readmitted to the receiver, the beam returns to the horizontal position.

This experiment shows that, although the brass counterpoises the cork in air, the true weight of the cork is greater than the true weight of the brass. The apparent weight in air of either body is the true weight diminished by the weight of the air which the body displaces. As the cork displaces more air than the brass, the upward pressure of the atmosphere on it is greater than the upward pressure on the brass.



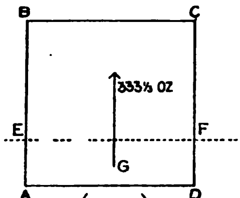
Baroscope.

Ex. 1.—Suppose the plane of the paper to be vertical; draw a square ABCD; take AE a third of AB and DF a third of the parallel side DC, and draw a line of indefinite length through EF; let that line represent the surface of water, and let the square represent a cube (whose edge is a foot long) held in it, with AD under water; find the resultant of the fluid pressure on the cube. If now we suppose the cube turned round E, so that D comes into the surface of the water, find the resultant pressure in this case. (See figs. page 134.)

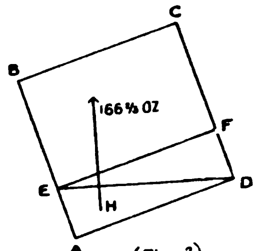
In the first position, in which EF is the surface of the liquid, the resultant pressure of the water on the cube is equal to the weight of water AEFD which is displaced by the body, and acts vertically upwards through G (fig. 1), the centre of gravity of AEFD. The volume of the part of the cube which is immersed is evidently 1/3 of the volume of the cube, that is, is 1/3 of a cubic foot. Hence the resultant fluid pressure on the cube in this position is equal to the weight of 1/3 of a cubic foot of water, and this

$$=\tfrac{1}{3}\times 1000, =333\tfrac{1}{3}\text{ oz.}$$

When the cube is turned into the position in which ED is horizontal, the part of the volume of the cube which is immersed



(Fig. 1)



(Fig 2)

= volume of EAD, = half volume of EADF,
= 1/6 of a cubic foot.

Hence the resultant pressure on the cube in this case

= weight of 1/6 of a cubic foot of water,
$= \frac{1}{6} \times 1000, = 166\frac{2}{3}$ oz.

This pressure acts upwards through the centre of gravity H (fig. 2) of EAD.

Ex. 2.—A piece of metal, whose specific gravity is 8, weighs 10 lbs. in air. What will be its apparent weight in alcohol, whose specific gravity is ·8?

Since the specific gravity of the metal is 8, it follows that the weight of a volume of water equal to the volume of the body is $\frac{1}{8} \times 10, = 5/4$ lbs. Hence the weight of the same volume of alcohol is $\frac{5}{4} \times {\cdot}8, = 1$ lb. When, therefore, the metal is weighed in alcohol, it is pressed upward with a force equal to the weight of 1 lb. Hence the apparent weight in alcohol is 9 lbs.

Ex. 3.—A body, which sinks in water, weighs a grammes in air and b grammes in water. Show that the volume of the body is $a - b$ cubic centimetres, and that its specific gravity is $a/(a-b)$.

The loss of weight in water is $a - b$ grammes, and this is equal to the weight of a volume of water equal to the volume of the body. But a cubic centimetre of water weighs one gramme, and therefore the volume of $a - b$ grammes of water is $a - b$ cubic centimetres. It follows that the volume of the body is $a - b$ cubic centimetres.

Also the specific gravity of the body is equal to the ratio of the weight of the body to the weight of an equal volume of water, and is therefore equal to the ratio of a to $a - b$, that is, is equal to the fraction $a/(a-b)$.

Ex. 4.—Assuming that atmospheric air at ordinary pressures and temperatures weighs ·31 grain, and hydrogen ·02 grain per cubic inch, what will be the weight of a bladder in which are 300 cubic inches of hydrogen, that will stay where it is placed, *e.g.*, on the table?

The weight of the bladder, together with the weight of hydrogen which it contains, must not be less than the weight of air which it displaced. Hence if w denote the least weight of the bladder, we get

$w+$ weight of 300 cubic inches of hydrogen
$=$ weight of 300 cubic inches of air,

or $w+300\times\cdot 02=300\times\cdot 31$;

from which $w=300\times\cdot 29=87$ grains.

EXAMPLES IX.

[A cubic foot of water weighs 1000 oz. Take $\pi=22/7$.]

(*The Answers are given on page* 335.)

A.

1. Find in pounds weight the resultant pressure on a vertical lock gate, 14 feet broad, against which water rises to a depth of $9\frac{1}{2}$ feet.

2. A rectangle ABCD is placed in a liquid, whose specific gravity is $1\cdot 5$, with the edge AB in the surface and the opposite edge CD 3 feet below the surface. If the lengths of the sides AB and AD are 4 feet and 5 feet respectively, determine in pounds weight the resultant pressure of the liquid on one side of the rectangle.

3. A rectangular tank is full of water, its length and depth being respectively 10 feet and 5 feet. A diagonal line is drawn along one of its vertical sides dividing it into two triangles. Find in pounds weight the magnitude of the resultant of the fluid pressures on each of these triangles. Why is the answer independent of the width of the reservoir?

4. A rectangular board is immersed in water with one of its edges parallel to the surface, and at a depth below the surface equal to the height of the rectangle. Compare the whole pressures on the board when it is (i) horizontal, (ii) vertical and upwards, (iii) vertical and downwards.

5. The depth of water in a vessel is 10 feet, the base being a circle of $1\frac{3}{4}$ feet radius. Find in pounds weight the pressure on the base of the vessel.

Why is the answer independent of the shape of the sides of the vessel?

6. An equilateral triangle, each side of which is 1 foot, is immersed, vertex downwards, in water with the base horizontal, and at a depth of 19 feet. If the plane of the triangle makes an angle of 30° with the vertical, find in pounds weight the whole pressure on one side of the triangle.

7. Find in pounds weight the *total* pressure on a cube whose edge is one foot, immersed to an average depth of 1000 fathoms in the ocean, a cubic foot of sea-water weighing 1032 oz.

What is the *resultant* pressure?

8. Determine the thrust in pounds weight on every foot length of a vertical wall of a rectangular reservoir, due to water 150 feet deep, and determine the depth at which the resultant pressure may be supposed to act.

9. Determine the thrust in pounds weight due to a depth of 100 feet of water on every foot length of a reservoir wall, inclined to the horizon at a

slope of 1 in $1\frac{1}{2}$, and determine the line along which the resultant pressure may be supposed to act.

10. A lock gate, 10 feet wide and 10 feet deep, has water on one side 8 feet deep and on the other 5 feet deep, in each case measured from the lower edge of the gate. Determine in pounds weight the resultant pressure on the lock gate and its point of application.

11. A cylindrical vessel, whose base is a circle of 4 inches radius, and whose height is 10 inches, is placed with its axis vertical, and is partly filled with mercury and partly with water, the volumes of the two liquids being equal.

Find in pounds weight (i) the pressure on the base, (ii) the *total* pressure on the sides of the vessel, taking the specific gravity of mercury to be 13·6.

Show that the *resultant* pressure on the sides of the vessel is zero.

12. A hollow sphere, whose radius is 6 inches, is just filled with equal volumes of mercury and water. Find in pounds weight (i) the *total* pressure on the surface of the sphere, (ii) the *resultant* pressure on the upper half of the surface of the sphere, (iii) the *resultant* pressure on the lower half of the surface of the sphere, (iv) the *resultant* pressure on the whole surface of the sphere.

13. A conical vessel with its base horizontal is filled partly with mercury and partly with water; internally the radius of the base is 6 inches, and the height 18 inches. If the volume of the mercury is seven times that of the water, find the pressure in pounds weight on the base, having given that a cubic inch of water weighs 250 grains, and a cubic inch of mercury 3400 grains.

Also find the resultant pressure in pounds weight on the sides of the vessel.

14. The mass of a wooden sphere, whose radius is 1 foot, is 250 lbs. If the sphere is totally immersed in water, and then left free to move, will it rise or sink in the water?

15. What is the apparent weight (i) in water, (ii) in mercury, of a piece of platinum whose mass is 105 grammes and specific gravity 21, the specific gravity of mercury being 13·6?

16. From the scale-pans of a balance are suspended a piece of gold and a piece of platinum respectively. The substances are immersed in water and there is equilibrium. Taking the specific gravities of gold and platinum to be 19·3 and 21 respectively, find the mass of the gold, the mass of the platinum being 210 grains.

17. If in the preceding question the platinum were immersed in alcohol, specific gravity ·8, the other conditions of the question being unaltered, what must be the mass of the gold?

18. An empty balloon, with its car and appendages, weighs *in air* 1200 lbs. If a cubic foot of air weighs $1\frac{1}{4}$ oz., how many cubic feet of a gas, whose density *relative to air* is ·52, must be introduced before the balloon will begin to ascend?

19. With the barometer at 760 mm., the mass of a litre of air is 1·3 grammes, and of a litre of hydrogen is ·089 gramme. The materials of a balloon weigh 50 kilogrammes; what must be its volume in order that it may just rise when filled with hydrogen?

B.

20. A cylindrical vessel, open at the top and containing water, is gradually tilted, none of the water being spilt, till the pressure on the base is diminished by a half. What is the inclination of the cylinder to the vertical?

The question plainly presupposes that the whole of the base of the cylinder remains under water. What, in terms of the radius of the base, is the smallest height of the cylinder and the smallest quantity of water for which this condition would be fulfilled?

21. A rectangular area ABCD is just immersed in water with the side AB in the surface. Find a point P in AB such that the pressure on the triangle APD may be one-fifth of the whole pressure on the rectangle.

22. A triangle has its base horizontal and its vertex in the surface of a liquid. Divide it by lines parallel to the base into three parts on which the pressures are equal.

23. A cylinder, with its axis vertical, is filled with liquid, and is divided by horizontal sections into 5 annuli, so that the total pressure on each is equal to the pressure on the base. If the radius of the base is 10 inches, determine the height of the cylinder and the breadth of the third annulus.

24. A triangle ABC, whose plane is vertical, is wholly immersed in water in such a manner that the angle A is in the surface, and the bisector of the angle B is horizontal, and meets AC in D. Show that if AB $= \frac{2}{5}$ of BC, the whole pressure on the triangle ABD is to the whole pressure on the triangle BCD as 9 is to 35.

25. A hollow prism of triangular section is filled with liquid, and is laid on a horizontal plane with the base of the triangle in contact with the plane. (i) Prove that the pressure on the base is twice the weight of the liquid. (ii) Prove that the pressure on one of the faces is to the pressure on the base as the area of that face is to twice the area of the base. (iii) Find the resultant of the pressures on the base and the two faces.

26. A conical vessel, 8 feet high, is filled with equal volumes of two liquids which do not mix. If the weight of a cubic foot of the denser liquid is w, and of the other is $\frac{1}{2}w$, find the pressure on the base, supposed to be horizontal, and to be 1 sq. ft. in area.

27. A vertical cylinder contains water to a depth equal to the diameter of the circular base. A sphere of four times the density of water and of the same radius as the cylinder is placed upon the liquid and fits the cylinder water-tight. Show that the pressure of water at its uppermost parts is now that due to a depth of water 7/3 of the diameter, and that the

increase of total pressure on the curved sides is 14 times the weight of water.

28. A right-angled triangle is just immersed in water with its hypotenuse vertical; the parts into which the triangle is divided by a horizontal line drawn through the right angle sustain equal fluid pressures. Show that the tangent of the smallest angle of the triangle is $2/\sqrt{3+\sqrt{17}}$.

29. ABCD is a rectangle immersed in a fluid with the side AB in the surface, and AC is one diagonal. Given that the depths of the centres of pressure of the whole rectangle and the triangle ABC are 2/3 and 1/2 of the depth of C respectively, find the depth of the centre of pressure of the triangle ADC.

30. Prove that the thrust on every foot length of a vertical reservoir wall, a feet high, due to the pressure of water is $w(ah+a^2/2)$ pounds weight, acting at a height $\frac{1}{3}a(a+3h)/(a+2h)$ above the foot of the wall, where w denotes the weight in lbs. of a cubic foot of water, and h the head of water due to atmospheric pressure.

31. A hollow cone, height h, radius of base r, is held with its axis vertical and vertex downwards, and is filled with liquid the weight of unit volume of which is w. Find the magnitudes of the resultant *vertical* pressure and resultant *horizontal* pressure respectively on one of the quarters into which the curved surface of the cone is divided by two planes at right angles drawn through the axis. By compounding these two forces find the magnitude of the resultant pressure.

32. A hollow sphere, radius r, is filled with liquid, the weight of unit volume of which is w. Find the magnitude of the resultant fluid pressure on one of the halves into which the surface of the sphere is divided by a vertical plane through its centre.

33. A vessel with a horizontal base, and sides of any form, contains liquid. (i) Show the resultant horizontal pressure on the sides is zero. (ii) Show that the pressure on the base is equal to the weight of a column of the liquid whose height is equal to the depth of the liquid, and whose sectional area is equal to the area of the base. (iii) Show in what cases the pressure on the base will be greater than, and in what cases less than, the weight of liquid in the vessel.

34. An imaginary closed surface of any form is drawn through a mass of heavy fluid. Show that the resultant horizontal pressure of the surrounding fluid on this surface is zero.

What is the magnitude and line of action of the resultant pressure on the surface?

35. A hemisphere of radius 1 foot is immersed in water with its base inclined at an angle of 60° to the horizon, and its centre is 6 feet deep. Find the magnitude of the resultant pressure of the water on the curved surface.

36. A closed conical vessel whose height is twice the diameter of its base is just filled with water, and suspended by a string attached to a point in the circumference of the base. Prove that if the weight of the vessel be

neglected, the vertical component of the pressure of the water on the curved surface will be 13/12 of the weight of water in the vessel.

37. A solid cone is immersed in water with its lowest generating line horizontal, and with the centre of its base at a depth c below the surface of the water. If r is the radius of the base of the cone, h the height, and w the weight of unit volume of water, find the horizontal and vertical components of the pressure on the curved surface. By compounding these two forces, find the magnitude of the resultant pressure on the curved surface.

38. A closed cylinder, the diameter of whose base is equal to its length, is full of water, and hangs freely by a string fastened to a point in its upper rim. Prove that, the weight of the cylinder being neglected, the vertical and horizontal components of the resultant pressure on its curved surface are each half the weight of water.

CHAPTER VIII.—DETERMINATION OF SPECIFIC GRAVITY BY EXPERIMENT.

81. Methods of determining Specific Gravity.

We have defined specific gravity of a substance to be the ratio of the density of the substance to the density of water, and we have shown (Art. 26) that the specific gravity is equal to the ratio of the mass (often called weight) of any volume of the substance to the mass of an equal volume of water.

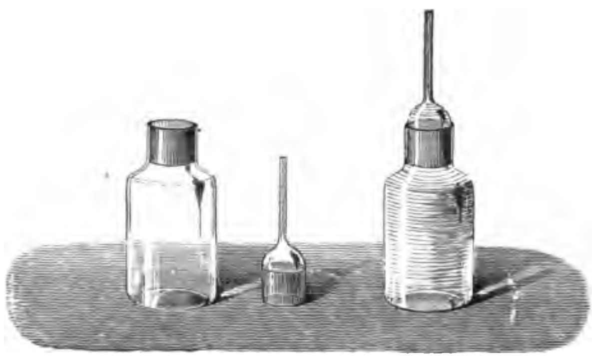
In all determinations of specific gravity it is therefore necessary to find (1) the mass of the whole volume or part of the volume of the substance, and (2) the mass of an equal volume of water. The mass of the substance may be found directly by the balance, so that the problem of finding the specific gravity is reduced to that of finding the mass of a volume of water equal to the volume of the substance.

The mass of water equal in volume to the volume of the substance is found by the **specific gravity bottle**, or by the **hydrostatic balance**. By the use of the specific gravity bottle the mass of a definite volume of the substance and the mass of the same volume of water are directly determined. The method of using the hydrostatic balance for determining the

mass of a volume of water equal to the volume of a substance is based on the principle of Archimedes (Art. 78).

The volumes of some solid bodies of regular shape can be calculated when certain lengths in the bodies have been measured. In such cases the absolute density of the body may be found by dividing the mass by the volume, and then the specific gravity is found by dividing the absolute density by the density of water.

In a similar way the specific gravity of a liquid or a gas



Specific Gravity Flask for Solids.

may be found by determining by experiment the mass of a measured volume of the liquid or gas.

For the practical determination of the specific gravities of liquids instruments of various forms, called **hydrometers**, are often used.

82. Use of the Specific Gravity Bottle.

The specific gravity bottle is a small glass bottle fitted with a stopper. It may be used to determine the specific gravity of a liquid, or of a solid broken into small pieces, or of a powder insoluble in water.

By the use of the specific gravity bottle and the ordinary balance we can find the mass of a definite volume of a liquid. For by finding the mass of the bottle, first when empty, and secondly when filled with the liquid, we can, by taking the

difference of these two masses, find the mass of the volume of liquid which fills the bottle. If great accuracy is required, certain precautions must be taken to prevent the presence of air bubbles. For these precautions the student must consult works on Practical Physics.

(i) *To find the specific gravity of a liquid.*

In finding the specific gravity of a liquid by means of the specific gravity bottle, we determine by the balance the following three masses:—(1) The mass of the bottle when empty, (2) the mass of the bottle when filled with water, (3) the mass of the bottle when filled with the liquid. Let W_0, W_1, W_2 denote respectively these masses.

Then $W_1 - W_0$ is the mass of the volume of water which fills the bottle,

and $W_2 - W_0$ is the mass of the same volume of the liquid.

Hence the specific gravity of the liquid is

$$(W_2 - W_0)/(W_1 - W_0).$$

(ii) *To find the specific gravity of a solid in small pieces, whose density is greater than that of water.*

We make the following determinations by means of the balance:—(1) We weigh the pieces of the solid in air; let the mass be W_0. (2) We weigh the specific bottle when filled with water; let the mass be W_1. (3) We put the pieces of the solid into the bottle, fill up the bottle with water, and weigh it; let the mass be W_2.

At the third weighing the solid displaces its own bulk of water. The quantity of water therefore in the bottle at the third weighing is less than the quantity at the second weighing by the amount of water displaced by the solid. It follows that

$W_2 - W_1$ is the *excess* of the mass of the solid over the mass of water which the solid displaces. Therefore

$$W_0 - (W_2 - W_1), \; = W_0 - W_2 + W_1,$$

is the mass of the water displaced by the solid, that is, is the

mass of water whose volume is equal to the volume of the solid. Hence the specific gravity of the solid is

$$W_0/(W_0 - W_2 + W_1).$$

In working all examples of the determination of specific gravities the student is recommended to obtain the result without quoting a formula.

Ex. 1.—A glass bottle is weighed empty, filled with water, and filled with nitric acid. The masses in the three cases are 43·2, 108·4, and 141 grammes respectively. Find the specific gravity of nitric acid.

Mass of water which fills the bottle

$$= 108·4 - 43·2, = 65·2 \text{ grammes.}$$

Mass of nitric acid which fills the bottle

$$= 141 - 43·2, = 97·8 \text{ grammes.}$$

Hence the specific gravity of nitric acid

$$= 97·8/65·2, = 1·5.$$

Ex. 2.—A specific gravity bottle full of water weighs 1100 grains. When a quantity of sand, which weighs 355 grains in air, is put into the bottle, and the water thus displaced removed, it is found that the bottle and its contents weigh 1205 grains. Find the specific gravity of the sand.

The excess of the mass of the sand over the mass of water which the solid displaces

$$= 1205 - 1100, = 105 \text{ grains;}$$

hence the mass of water whose volume is equal to the volume of the sand

$$= 355 - 105 = 250 \text{ grains,}$$

and the specific gravity of the sand is

$$= 355/250 = 1·42.$$

83. Use of the Hydrostatic Balance.

The use of the hydrostatic balance depends on the principle (Art. 78) that when a body is weighed in a fluid, the apparent loss in the weight of the body is equal to the weight of fluid which the body displaces. The hydrostatic balance may be used to determine the specific gravity of a liquid or of a solid.

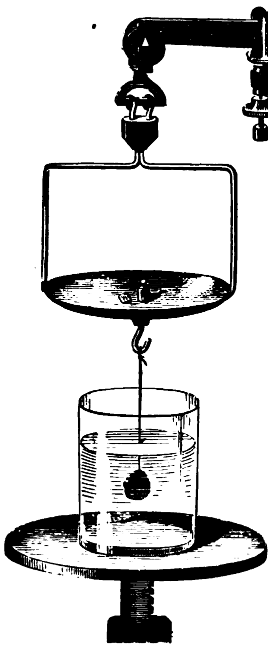
(i) *To determine the specific gravity of a liquid.*

We take a solid which sinks both in water and in the liquid, and make the following determinations:—(1) We weigh the

solid in air; let W be its weight. (2) We weigh the solid in water; let W_1 be its apparent weight. (3) We weigh the solid in the liquid; let W_2 be its apparent weight. Then

$W - W_1$ is the apparent loss of weight in water,

and therefore represents the weight of water whose volume is equal to the volume of the solid; and



Specific Gravity of Solids.

Specific Gravity of Liquids.

$W - W_2$ is, in the same way, the weight of the same volume of the liquid.

Hence the specific gravity of the liquid is

$$(W - W_2)/(W - W_1).$$

(ii) *To determine the specific gravity of a solid denser than water.* In this case it is only necessary to determine the weight of the body in air, W say, and the apparent weight of the body in water, W' say. Then

$W - W'$ represents the weight of water whose volume is equal to the volume of the body.

Hence the specific gravity of the body is

$$W/(W - W').$$

(iii) *To determine the specific gravity of a solid less dense than water.*

A solid which is less dense than water floats in water, and therefore its apparent weight in water is zero. In this case the specific gravity may be found by attaching to the body a piece of some dense material such as lead, called the **sinker**, which will make the body sink in water.

Let W denote the weight of the body in air, y the apparent weight of the sinker in water, and W' the apparent weight of the body and sinker together in water. Then the specific gravity of the body is

$$W/(W - W' + y).$$

To prove this, let us denote for a moment the weight of the sinker in air by x. Then

$W + x - W' =$ weight of water displaced by body and sinker,
and $x - y =$ weight of water displaced by the sinker alone.

Hence, by subtraction, we get the result—

$W - W' + y =$ weight of water displaced by body alone.

Dividing W, the weight of the body, by the last expression, we get, as the specific gravity of the body,

$$W/(W - W' + y).$$

We may arrive at this result in another way:—W' is the weight of the body and sinker together in water, and y is the weight of the sinker alone in water; therefore $y - W'$ is the *excess* of the upward pressure of the water on the body over the weight of the body, that is, is the excess of the weight of water displaced by the body over the weight of the body. Hence $W + (y - W'), = W - W' + y$, is the weight of water displaced by the body, and the result follows as before.

(iv) *To determine the specific gravity of a solid soluble in water.*

We weigh the body in air and in a liquid, of known specific gravity, in which the body is not soluble.

Let W denote the weight of the body in air, and W' the apparent weight in the liquid, and s the specific gravity of the liquid.

Then $W - W'$ is the weight of a volume of the liquid equal to the volume of the body, and therefore

$$\text{Specific gravity of body : specific gravity of liquid} = W : W - W',$$

from which we get

$$\text{Specific gravity of body} = sW/(W - W').$$

Ex. 1.—A body weighs 150 grains in air, and, when immersed in water by means of a sinker, the combination weighs 45 grains. The sinker alone weighs 75 grains in water. Find the specific gravity of the body.

The excess of the weight of water whose volume is equal to the volume of the body over the weight of the body is

$$= 75 - 45, = 30 \text{ grains}.$$

Hence the weight of water displaced by the body

$$= 150 + 30, = 180 \text{ grains},$$

and the specific gravity of the body is 150/180, $= 5/6$.

Ex. 2.—A solid weighs 30 grammes in water, and 40 grammes in a liquid whose specific gravity is ·8. What is the volume of the body, and what is its mass?

Let W denote the number of grammes in the body. Then

$W - 30 =$ weight in grammes of water which body displaces,

and

$W - 40 =$ weight in grammes of liquid of sp. gr. ·8 which body displaces.

Hence

$$(W - 40)/(W - 30) = {\cdot}8,$$

from which we get

$$5(W - 40) = 4(W - 30),$$

giving

$$W = 80 \text{ grammes}.$$

Also the weight of water displaced by body is $W - 30$, $= 50$ grammes. But the volume of 1 gramme of water is 1 cubic centimetre. Hence the volume of water displaced by the body, that is, the volume of the body, is 50 cubic centimetres.

84. Corrections for Weighing in Air.

We saw (Art. 80) that when the mass of a body is determined by weigh-

ing it in air against a standard mass, the result obtained is not the true mass of the body. In order to obtain the true mass directly it would be necessary to weigh the body in vacuo. We may, however, arrive at the true mass by applying to the result of weighing the body in air a correction for the weights of air displaced by the body and by the standard mass which counterpoises it in air. In making this correction we use the following **exact equation** connecting the true weight of the body and the true weight of the standard mass which counterpoises it in air:—

True weight of body − weight of air displaced by body
= weight of standard mass − weight of air displaced by standard mass.

Let W_0 denote the true weight of the body, and W the apparent weight in air, so that W represents the true weight of the standard mass which counterpoises the body in air. Let k represent the ratio of the density of air to that of the body, and k' the ratio of the density of air to that of the standard mass. Then

weight of air displaced by body $= kW_0$,
and weight of air displaced by standard mass $= k'W$.

Hence from the above equation we get

$$W_0 - kW_0 = W - Wk',$$

from which
$$W_0 = W(1 - k')/(1 - k),$$

a formula from which W_0, the true weight of the body, may be calculated when W, k and k' are known.

Standard masses are usually composed of dense substances such as platinum or brass, so that k' is usually very small. We may therefore obtain an approximate value of W_0 by putting $k' = 0$. The result is the formula

$$W_0 = W/(1 - k).$$

Ex.—A piece of cork, whose specific gravity is ·24, is counterpoised by a standard brass mass of 4085 grains, whose specific gravity is 8. Assuming that, under the circumstances of the weighing, water is 817 times as dense as air, find the true mass of the cork.

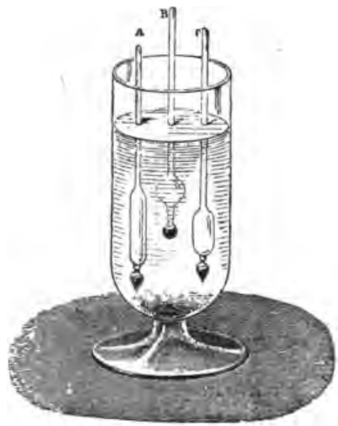
Here we get
$$W_0\left[1 - \frac{1/817}{\cdot 24}\right] = 4085\left[1 - \frac{1/817}{8}\right],$$

from which
$$W_0 = 4105{\cdot}3 \text{ grains}.$$

85. The Common Hydrometer.

Of instruments for determining specific gravities there are two classes:—(1) Hydrometers of **variable immersion** and **constant weight**, (2) hydrometers of **constant immersion** and **variable weight**.

In the accompanying figure are shown forms of the common hydrometer of variable immersion, an instrument which is used for determining the specific gravity of a liquid. It consists of a long slender stem, fitted to a hollow cylinder or ball of glass, terminating in a bulb filled with mercury. When the instrument is placed in a liquid, it floats in a vertical position with the weighted end and part of the stem under the surface of the liquid.



Hydrometers.

When the hydrometer floats in a liquid, the point of the stem which is in the surface depends on the specific gravity of the liquid. For the weight of the hydrometer is constant and equal in all cases to the weight of the liquid displaced. Hence the denser the liquid, the less must be the volume displaced, that is, the denser the liquid, the shorter will be the part of the stem under the surface.

The stem is graduated by the maker in such a way that, when the instrument floats in a liquid, the reading on the stem at the point to which the instrument sinks gives the specific gravity of the liquid.

Ex.—A common hydrometer floats in a liquid, whose specific gravity is ·9, with 5 inches more of its stem below the surface than it would have if placed in water. What point of the stem will be in the surface when it floats in a liquid whose specific gravity is ·95?

Let V denote the number of cubic inches of water displaced by the hydrometer when floating in water, and a the number of square inches in the area, taken to be uniform, of the section of the stem. Then when the hydrometer floats in the liquid whose specific gravity is ·9, the volume of the liquid displaced is $(V+5a)$ cubic inches. Hence, since the weights of water and liquid displaced are equal, we get

$$\tfrac{9}{10}(V+5a)=V,$$

from which $$V=45a.$$

When the hydrometer floats in the liquid, whose specific gravity is ·95, let x represent in inches the height of the point of the stem which is in the surface above the point which is in the surface when the hydrometer floats in water. Then since the weight of the liquid displaced is equal to the weight of V cubic inches of water,

$$\tfrac{19}{20}(V+xa)=V.$$

Putting in this equation $\qquad V=45a,$

we get $\qquad 19(45a+xa)=45a\times 20;$

therefore $\qquad 19x=45,$

or $\qquad x=2\tfrac{7}{19}.$

Hence, when the hydrometer is placed in a liquid of specific gravity ·95, it will float with $2\tfrac{7}{19}$ inches more of its stem under the surface than when it is placed in water.

86. Graduation of the Common Hydrometer.

A formula may be investigated by which the point on the stem to which the instrument sinks in a given liquid may be determined.

Let A be the point on the stem which is in the surface when the hydrometer floats in water, and let V denote the volume of the bulb and the part of the stem below A. Let a denote the area of the stem, taken to be uniform, and let w denote the weight of unit volume of water. It is required to determine the point of the stem which is in the surface when the instrument floats in a liquid whose specific gravity is s.

First let s be less than unity; then the point B to which the instrument sinks is above A. Let the length of BA be denoted by x, and let w denote the weight of unit volume of water. Then, by the principle of Archimedes, the weight of the instrument is equal to wV, and also equal to $(V+ax)sw$.

Hence $\qquad (V+ax)sw=Vw,$

from which $\qquad x=V(1/s-1)/a,$

an equation which determines AB.

Next let s be greater than unity, and let C be the point to which the instrument sinks in the liquid; then C is below A. If x denotes AC, we

get $\qquad (V-ax)sw=Vw,$

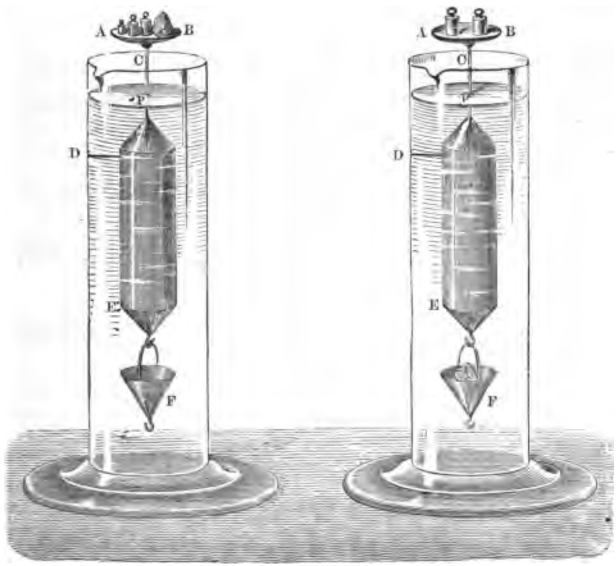
from which $\qquad x=V(1-1/s)/a,$

an equation which determines AC.

It is found convenient in practice to use two classes of hydrometers. One class is used for determining the specific gravities of liquids less dense than water. In an instrument of this class, the point A to which the liquid sinks in water is near the lower end of the stem. The part of the

stem above A is graduated, the point A being marked 1, and the readings decreasing from A upwards. In hydrometers of the other class, which are used for determining the specific gravity of liquids denser than water, the point A to which the instrument sinks in water is near the top of the stem. The part of the stem below A is graduated, the point A being marked 1, and the reading increasing from A downwards.

Ex.—If V denote the *whole* volume of an hydrometer, s_1 and s_2 the specific gravities of two liquids in which the instrument floats with the lengths



Nicholson's Hydrometer.

x_1 and x_2 respectively above the surface, and if a denote the area of the section of the stem, then $s_1/s_2 = (V - ax_2)/(V - ax_1)$.

For if w denote the weight of unit volume of water, the weight of the instrument is $s_1 w (V - ax_1)$, and is also equal to $s_2 w (V - ax_2)$. Equating these expressions, the result immediately follows.

87. Nicholson's Hydrometer.

Nicholson's Hydrometer is a form of an hydrometer of constant immersion, which is sometimes used in laboratories for determining the specific gravities of liquids and of small solids.

This instrument consists of a hollow brass cylinder DE, to one end of which is fastened a very thin stem of steel C, sup-

porting a pan AB. An iron stirrup, fixed to the other end of the cylinder, supports another pan F. A fine mark is cut on the stem at some point P.

(*a*) *To find the specific gravity of a liquid by Nicholson's hydrometer.*

Let W denote the weight of the instrument.

Let the instrument float in water, and let W_1 denote the weight which must be placed in the pan AB to make the instrument sink to the mark P.

Let the instrument float in the liquid, and let W_2 denote the weight which must be placed in AB to sink the instrument to the mark.

Then

$W + W_1$ is the weight of water displaced by the instrument,

and $W + W_2$ is the weight of the liquid displaced by the instrument.

Since the volume is the same in the two cases, the specific gravity of the liquid is

$$(W + W_2)/(W + W_1).$$

(*b*) *To find the specific gravity of a small solid by Nicholson's hydrometer.*

Let the instrument float in water, and let W denote the weight which must be placed in the pan AB to make the instrument sink to the mark.

Remove the weight W, and let the solid be placed in the pan AB as in the left-hand figure. Let W_1 be the weight which must now be placed in AB to make the instrument sink to the mark.

Remove the solid and place it in the lower pan F, as in the right-hand figure. Let W_2 denote the weight which must now be placed in the pan AB to make the instrument sink to the mark.

Then the weight of the solid in air is

$$W - W_1,$$

and the apparent weight of the solid in water is

$$W - W_2.$$

Hence the weight of an equal volume of water is $(W - W_1) - (W - W_2)$ or $W_2 - W_1$, and the specific gravity of the solid is

$$(W - W_1)/(W_2 - W_1).$$

Ex.—It is found that the hydrometer sinks to the mark when a mass of 12 oz. is placed in the pan. A solid is placed in the upper pan, and it is found that the mass now required to sink the instrument to the mark is 5 oz. The solid is placed in the lower pan, and it is found that the instrument sinks to the mark when a mass of 7 oz. is placed in the pan. Find the specific gravity of the solid.

Here

mass of solid $= 12 - 5, = 7$ oz.;
apparent weight of solid in water $= 12 - 7, = 5$ oz.;
mass of water displaced $= 7 - 5, = 2$ oz.

Hence the specific gravity of the solid is $7/2, = 3{\cdot}5$.

EXAMPLES X.

(*The Answers are given on page* 335.)

1. A solid weighs 12 oz. in air, 8 oz. in water, and 9 oz. in a certain liquid. Find the specific gravities of the solid and of the liquid.

2. A quantity of pebbles weighing $111\frac{1}{4}$ lbs. is put into a vessel of 1 cubic foot capacity, which then requires 540 cubic inches of water to fill it completely. Find the weight of a cubic foot of the stone of which the pebbles consist, and also find the specific gravity of the stone.

3. A solid weighs 120 grammes in air and 90 grammes in a liquid of specific gravity ·9. Find the specific gravity of the solid.

4. The weight of a piece of wood is 4 lbs. and of a piece of lead in water 4 lbs., and of the lead and wood in water 3 lbs. What is the specific gravity of the wood?

5. Find the specific gravity of a piece of cork from the following data:—

Weight of cork in air $= 2$ grammes.
Weight of cork and sinker in water $= 4$ grammes.
Weight of sinker in water $= 12$ grammes.

6. A cork weighs in air 15 grains. The sinker weighs in water 90 grains, and the cork and sinker together weigh in water 45 grains. Required the specific gravity of the cork.

7. A substance of which the specific gravity is 6 weighs 180 grains in air and 155 in a liquid. Required the specific gravity of this liquid.

8. A solid weighs 100 grains in water and 115 in a liquid of which the specific gravity is ·8. What is the weight of the solid?

9. The weight of a Nicholson's hydrometer is 8 oz. The weight required to sink it to the mark in water is $3\frac{1}{8}$ oz., and in alcohol is 1 oz. Find the specific gravity of alcohol.

10. The weight of a Nicholson's hydrometer is 10 oz. The weight required to sink it to the mark in a liquid whose specific gravity is 1·2 is 4 oz. What weight will be required to sink it to the mark in water?

11. The weight required to sink a Nicholson's hydrometer in water to the mark is 4·2 oz.; when a small body is put in the upper pan the weight required is 1·26 oz., and when the body is put in the lower pan the weight required is 2·73 oz. Find the specific gravity of the body.

12. The standard weight in a Nicholson's hydrometer being 1200 grains, a body is placed in the upper pan, and it is found that 200 grains must be added to sink the hydrometer to the standard point. The body is now placed in the lower pan, and it is found that 450 grains must be placed in the upper pan to sink the instrument to the standard point. What is the specific gravity of the body?

13. The standard weight in a Nicholson's hydrometer is 1250 grains; a small substance is placed in the upper pan, and it is found that 530 grains are required to sink the instrument to the standard point, but when the substance is put into the lower pan 620 grains are required. What is the specific gravity of the substance?

14. A Nicholson's hydrometer weighs 12 oz. The addition of 3 oz. to the upper pan causes it to be sunk in one liquid to the marked point, while $7\frac{1}{2}$ oz. are required to produce the like result in another liquid. Compare the specific gravities of the liquids.

15. The apparent weight of a piece of platinum in water is 60 grammes, and the absolute weight of another piece of platinum twice as big as the former is 126 grammes. Determine the specific gravity of platinum.

16. A piece of wax weighs $4\frac{1}{2}$ grammes in air. A piece of platinum, whose volume is one-tenth of a cubic centimetre and specific gravity 21, is attached to the wax, and the two together weigh in water $2\frac{1}{2}$ grammes. Find the specific gravity of the wax.

17. A piece of lead weighs 50 grammes in air, and when suspended in a liquid whose density is 1·2 grammes per cubic centimetre it weighs 44·6 grammes. Determine the density of the lead.

18. A specific-gravity bottle full of water weighs 44 grammes, and when some pieces of iron, weighing 10 grammes in air, are introduced into the bottle and the bottle is again filled up with water, the combined weight is 52·7 grammes. What is the specific gravity of the iron?

19. A body of density ·8 weighing 100 grammes in air is attached to a lead sinker weighing 50 grammes in water, and the two are completely immersed in water. What weight is required in the opposite scale for equilibrium?

20. An iron shell is found to lose half its weight when weighed in water; what part of its volume is hollow? (specific gravity of iron = 7·2).

21. The stem of a common hydrometer is cylindrical, and the highest graduation corresponds to a specific gravity of 1, while the lowest corresponds to a specific gravity of 1·2. What specific gravity corresponds to the point which is exactly midway between these two divisions?

22. An hydrometer is found to sink to points marked A, B, C on its stem when placed in water and in two other liquids (β and γ say) respectively, the specific gravity of γ being ·9. If B is half-way between A and C, what is the specific gravity of the liquid β?

23. A body weighs a grammes in water and b grammes in a liquid of specific gravity s. Find its true weight and volume.

24. If the same body have weights W_0, W_1, W_2,... in vacuo and in various liquids respectively, compare the specific gravities of these liquids.

25. Pieces of ice are floating in water, and gradually melt. Will the surface of the water be affected?

26. A piece of wood and a piece of brass exactly counterbalance each other in a perfectly just balance, the weighing being made in air. If the mass of the piece of brass is 1450 grains, what is the mass of the wood, given that the specific gravities of the wood and brass are ·85 and 8 respectively, and that 800 cubic inches of air weigh as much as 1 cubic inch of water at standard temperature?

CHAPTER IX.—FLOATING BODIES.

Bodies floating freely, Arts. 88 to 90.

88. Conditions of Equilibrium of a Body Immersed in a Fluid.

When a body is wholly or partly immersed in a fluid, the resultant action of the fluid on the body is a force equal to the weight of the fluid displaced, acting vertically upwards through the centre of gravity of the fluid displaced. The weight of the body is a force acting vertically downwards through the centre of gravity of the body. If the body is perfectly free to move, it will be in equilibrium only when the resultant fluid pressure is a force equal and opposite to the weight of the body.

Hence the following are the conditions that a body shall float freely in a liquid:—

(1) The weight of the body must be equal to the weight of the fluid displaced.

(2) The centre of gravity of the body and the centre of gravity of the fluid displaced must be in the same vertical line.

Also it is evident that, if the weight of the body is *less than the weight of the fluid displaced*, the body will *rise* in the fluid; and if *greater*, the body will *sink* in the fluid. Thus a balloon rises in the air because the total weight of the balloon is less than the weight of air displaced, and for a similar reason a bubble of air rises in water. On the other hand, a piece of metal, such as iron or lead, sinks in water because its weight is greater than the weight of water which it displaces.

It follows that when a body, ABCD (see upper figure, page 131), whose weight is less than the weight of an equal volume of any liquid, is placed in the liquid, the body will float in a position in which part only of its bulk, ABC, is below the surface of the liquid. In this position the weight of the body is equal to the weight of the part ABC of the liquid displaced.

Again, if the weight of a body, wholly immersed in a liquid, is just equal to the weight of its own volume of the liquid, the body will float at any depth.

In the case of a body of uniform density, these results may be expressed in the following way:—If the specific gravity of a body, wholly immersed in a liquid, is greater than the specific gravity of the liquid, the body will sink in the liquid; if the specific gravities of the body and of the liquid are equal, the body will float at any depth; and if the specific gravity of the body is less than the specific gravity of the liquid, the body will rise in the liquid, and will come to rest in a position in which part of its volume is immersed in the liquid, and the remainder above the surface.

89. Body Floating with a Part Immersed in each of Two Fluids.

When a body is placed with part of its volume immersed in one fluid, and the remainder in another fluid, each of the fluids exerts an upward force on the body. The amount of force

exerted by either fluid is equal to the weight of the volume of fluid displaced, and acts through the centre of gravity of this volume.

It follows that when a body floats freely with part of its volume immersed in each of two fluids, the weight of the body must be equal to the sum of the weights of the fluids displaced, and the centre of gravity of the body must be in the same vertical line with the point which is the centre of gravity of the two fluids displaced.

A similar proposition holds in the case in which a body floats freely with part of its volume immersed in each of more than two fluids.

A body floating partly immersed in water, and freely exposed to the atmosphere, is an example of a body floating immersed partly in water and partly in air. In this case the weight of the body is equal to the sum of the weights of water and air displaced.

Since the weight of any volume of air is very small compared with the weight of the same volume of water, the part of the weight of a solid supported by the upward pressure of the air is very small compared with the part supported by the upward pressure of the water. In the calculations of this chapter we shall neglect the upward pressure of the air on a solid floating partly immersed in a liquid.

90. Ratio of Part Immersed to Whole Volume.

A body of known specific gravity floats partly immersed in a liquid of known specific gravity. To find what part of the volume of the body is immersed in the liquid.

The body floats in the position in which its weight is equal to the weight of the liquid displaced.

Let V denote the whole volume of a body, s the specific gravity of the body, supposed to be of uniform density, V_1 the volume of the part immersed, and s_1 the specific gravity of the liquid. Also let w denote the weight of unit volume of water.

The weight of the body is wVs, and the weight of the liquid displaced is wV_1s_1; and these are equal.

Hence $wV_1s_1 = wVs$;

therefore $V_1s_1 = Vs$,

or $V_1/V = s/s_1$.

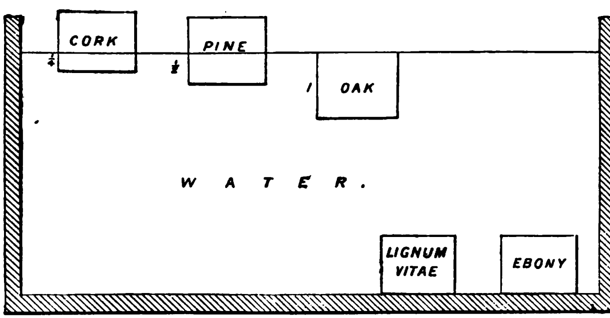
If W denote the mass of the body, and W_1 the mass of the part of the *body* below the surface, then W_1/W is equal to V_1/V, and therefore

$$W_1/W = V_1/V = s/s_1.$$

Cor.—If the liquid is water, $s_1 = 1$, and the equation becomes

$$V_1 = Vs.$$

Thus, the specific gravities of cork, pine, and oak being $\frac{1}{4}$, $\frac{1}{2}$,



and 1 respectively, it follows that a piece of cork will float in water with a quarter of its bulk immersed, a piece of pine with half of its bulk immersed, and a piece of oak with the whole of its volume immersed. Pieces of ebony and lignum-vitae, whose specific gravities are each greater than unity, would sink in water.

Ex. 1.—An iceberg is floating in sea-water, and it is estimated that the mass of ice above the surface is 10000 tons. Taking the specific gravities of ice and sea-water to be respectively ·92 and 1·026, find the total mass of the iceberg.

Taking the formula

$$W_1/W = s/s_1,$$

let W represent in tons the total mass of the iceberg, and W_1 the mass of

the part immersed. We have here $s = \cdot 92$, $s_1 = 1\cdot 026$, $W - W_1 = 10000$. Hence

$$W_1 = W \times \cdot 92/1\cdot 026,$$

and

$$W_1 = W - 10000.$$

Equating these expressions for W_1, we get

$$W \times \cdot 92/1\cdot 026 = W - 10000,$$

or

$$W \times \cdot 92 = W \times 1\cdot 026 - 10260.$$

Hence

$$W = 10260/(1\cdot 026 - \cdot 92),$$
$$= 96792 \text{ tons.}$$

Ex. 2.—The specific gravity of lead is 11·4, and that of cork is ·24. If a piece of cork weighs 3 lbs., find how much lead must be attached to it to make it sink in water.

Let x represent the least number of pounds of lead required. Then the sum of the weights of water displaced by the lead and the cork must just be equal to $(x+3)$ lbs.

The weight of water displaced by the lead is $x/11\cdot 4$ lbs., and the weight of water displaced by the cork is $3/\cdot 24$ lbs.

Hence

$$x + 3 = x/11\cdot 4 + 3/\cdot 24.$$

Solving this equation, we find

$$x = 10\cdot 4 \text{ lbs.}$$

Hence a piece of lead which weighs more than 10·4 lbs. would, when attached to the cork, make the cork sink in water.

Ex. 3.—A cube whose edge is a, and sp. gr. s, floats in a liquid whose sp. gr. is ns; another liquid, which does not mix with the former, and whose sp. gr. is s/n, is poured on the former liquid. What depth of the second liquid will be just sufficient to cover the cube?

Let x be the depth required. Then the cube floats with a vertical edge immersed to a depth $a - x$ in the liquid whose sp. gr. is ns, and the remainder, x, of the edge in the liquid whose sp. gr. is s/n. The volumes of the two liquids displaced are proportional to these lengths. The weights of the volumes of the liquids displaced and of the cube are proportional to $(a-x)ns$, $x \, . \, s/n$, as respectively.

By the principle of Archimedes,

$$as = (a-x)\,ns + xs/n,$$

from which $x = an/(n+1)$, the required depth of the second liquid.

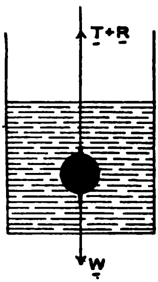
Bodies floating partly supported, Arts. 91 to 93.

91. Tension of the Thread supporting a Body Immersed in a Liquid.

A body, which would sink in a liquid, is wholly or partly

immersed in the liquid, and is supported by means of a thread. It is required to find the tension of the thread.

Let W denote the weight of the body, R the resultant pressure of the liquid on the body, and T the tension of the thread.



Then

$$T+R=W,$$

and therefore

$$T=W-R,$$

which gives a general expression for T.

(i) *Let the body be wholly immersed.*

Let s and s_1 denote the specific gravities of the body, supposed to be of uniform density, and of the liquid respectively, then

$$R/W=s_1/s.$$

Hence in this case

$$T=W-Ws_1/s,$$
$$=W(s-s_1)/s \qquad (1).$$

(ii) *Let the body be partly immersed.*

Let V and V_1 denote the whole volume of the body and the volume immersed respectively.

Then

$$R/W=V_1s_1/Vs,$$

giving

$$T=W-WV_1s_1/Vs$$
$$=W(Vs-V_1s_1)/Vs \qquad (2)$$

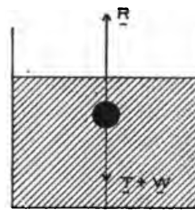
Cor. If the body, instead of being supported in the liquid by a thread, rests on the base of the vessel containing the liquid, the formula (1) or the formula (2) will give the pressure exerted by the body on the base of the vessel, according as the body is wholly or partly immersed.

92. Tension of the Thread which keeps a Body, Immersed in a Liquid, from Rising in the Liquid.

A body, which would rise in a liquid, is wholly or partly immersed in the liquid, and is kept from rising by means of

a thread fastened to a point below the surface. It is required to find the tension of the thread.

Using the notation of the preceding Article, we see that the body is kept at rest by the downward forces T and W and the upward force R.



Hence

$$T + W = R,$$

and therefore

$$T = R - W.$$

(i) *Let the body be wholly immersed.*

Then

$$R/W = s_1/s,$$

and therefore

$$T = Ws_1/s - W,$$

$$= W(s_1 - s)/s \qquad (1)$$

(ii) *Let the body be partly immersed.*

Then

$$R/W = V_1 s_1/Vs,$$

and therefore

$$T = W(V_1 s_1 - Vs)/Vs \qquad (2)$$

Cor. 1.—The value of T in formula (2) will be zero if $V_1 s_1 = Vs$. This agrees with the result of Art. 90.

Cor. 2.—The formula for T in equation (1) is the formula for the force required to submerge in a liquid a body whose specific gravity is less than that of a liquid.

93. Conditions of Equilibrium of a Solid Floating Partly Immersed in a Liquid, and Free to Turn about a Fixed Point in all directions.

The body is acted on by three forces—its weight, the upward pressure on it of the liquid, and the reaction of the fixed point. The weight of the body and the upward pressure of the liquid both act vertically, and therefore, by the theory of parallel forces, the reaction of the fixed point on the body must also act vertically, and the three forces must satisfy the conditions of equilibrium of three parallel forces. Hence the following are the conditions of equilibrium:—

(i) The centres of gravity of the body and of the liquid displaced must be in the same vertical plane through the fixed point.

(ii) The moments about the fixed point of the weight of the body and of the weight of the liquid displaced must be equal (Art. 37).

If a body floats partly immersed in a liquid, and if it is partly supported by a string, the string must be vertical. Also conditions (i) and (ii) must hold, the expression *the fixed point* being changed into *the point of attachment of the string to the body.*

Ex. 1.—A body whose mass is 12 lbs. and specific gravity 2·5 is placed in a vessel, with a horizontal base, containing water. What pressure is exerted on the base by the body, supposing the body to be completely covered with water?

The resultant pressure of the water on the body is a force acting vertically upwards equal to

$$12/2{\cdot}5, =24/5, =4\tfrac{4}{5} \text{ lbwt.}$$

Hence the pressure of the body on the base of the vessel is

$$12-4\tfrac{4}{5}, =7\tfrac{1}{5} \text{ lbwt.}$$

Ex. 2.—An ounce of silver, totally immersed in water, is supported by a string. If the water and the vessel which contains it originally weighed 30 ounces, what mass will counterbalance them after the ounce of silver is suspended in the water, and what is the tension of the string?

The upward pressure of the water on the ounce of silver

$$=1/10{\cdot}5, =2/21 \text{ ozwt.}$$

Hence the tension of the string

$$=1-2/21, =19/21 \text{ ozwt.}$$

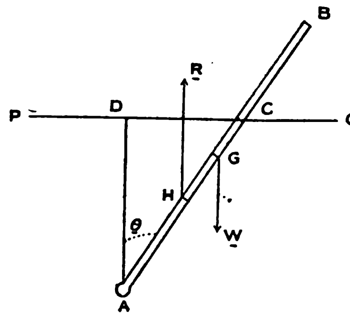
The silver exerts on the water a force equal and opposite to the force exerted by the water on the silver. Hence the weight of the vessel and water is apparently increased by the presence of the silver by the amount 2/21 oz. Thus after the silver is suspended in the water, the mass required to counterpoise the vessel and water is $30\frac{2}{21}$ oz.

Ex. 3.—A uniform rod AB can turn about a smooth hinge at A, which is immersed at a depth AB/2 below the surface of a liquid whose specific gravity is three times that of the rod. Find the position of stable equilibrium.

Let the figure represent the position of stable equilibrium, the angle

which AB makes with the vertical AD in this position being denoted by θ; and let C be the point of the rod which is in the surface PQ of the liquid.

The forces keeping the rod in equilibrium are (i) the weight, W say, of the rod, acting vertically downwards at G, the middle point of AB; (ii) the resultant pressure of the liquid on the rod, R say, acting vertically upwards at H, the middle point of AC; and (iii) the reaction at the hinge A. Since the rod is in equilibrium the moments about A of the forces W and R must be equal.



If s denotes the specific gravity of the rod, then $3s$ is the specific gravity of the liquid. Let A denote the sectional area of the rod, and w the weight of unit volume of water. Then

$$W = \mathrm{AB} \,.\, A \,.\, s \,.\, w, \text{ and } R = \mathrm{AC} \,.\, A \,.\, 3s \,.\, w,$$

and the moments of W and R about the point A are

$$W \,.\, \mathrm{AG} \sin\theta, = \tfrac{1}{2}\mathrm{AB}^2 \,.\, A \,.\, s \,.\, w \,.\, \sin\theta,$$

and

$$R \,.\, \mathrm{AH} \sin\theta, = \tfrac{1}{2}\mathrm{AC}^2 \,.\, A \,.\, 3s \,.\, w \,.\, \sin\theta,$$

respectively. Equating these moments, and cancelling factors common to both sides of the resulting equation, we get

$$\mathrm{AB}^2 = 3\mathrm{AC}^2.$$

But

$$\mathrm{AC} = \mathrm{AD}/\cos\theta, = \mathrm{AB}/2\cos\theta, \quad \text{(by conditions of the problem)}$$

and therefore

$$\mathrm{AB}^2 = 3\mathrm{AB}^2/4\cos^2\theta,$$

giving

$$\cos^2\theta = 3/4, \text{ and } \cos\theta = \sqrt{3}/2.$$

Hence θ, the angle BAD, is an angle of 30°.

EXAMPLES XI.

(The Answers are given on page 336.)

A.

1. A cube whose specific gravity is ·75 floats in water with four edges vertical; how much of one of its vertical edges is under water?

If its edges are 2 feet long, what is the magnitude of the resultant fluid pressure on the immersed part of one of its vertical faces?

2. A cylinder is 2 feet high, and the radius of its base is 3 feet; its specific gravity is ·7; it floats in water with its axis vertical. Find (i) how much of its axis will be under water, (ii) the force required to raise it one inch, (iii) the force required to depress it one inch.

3. A uniform cylinder floats in water with 1 inch of its length above the surface. In a liquid whose specific gravity is 1·5 it has 3 inches above the surface. What is the length of the cylinder? What is its specific gravity?

4. A body, which can just float when wholly immersed in water, floats in another liquid with 1/11 of its volume above the surface. What is the specific gravity of the liquid?

5. A body floats with 1/10 of its volume above the surface of water. What fraction of its volume will project above the surface if it were floating in a liquid of specific gravity 1·25?

6. A piece of iron, weighing 275 grammes, floats in mercury of specific gravity 13·59 with 5/9 of its volume immersed. Determine the volume of the iron in cubic centimetres and its density in grammes per cubic centimetre.

7. The weight of a cubical box is 1/4 of that of the water which fills it. When half filled with water, it sinks in water to a depth of 10 inches. Find the depth to which it sinks when filled with water.

8. A common hydrometer, floating in a liquid whose specific gravity is 1·1, has 5 inches of its stem above the surface. When it floats in a liquid whose specific gravity is 1·2 it has 6 inches of its stem above the surface. How much of the stem will be above the surface when it floats in a liquid whose specific gravity is 1·3?

9. A cylinder of wood, weighing 12 lbs., floats in water with 3/4 of its bulk immersed. When 5 lbs. of metal are tied to it, the two together can just float. Find the specific gravities of the wood and of the metal.

10. A cubic inch of lead of specific gravity 11·4 is attached to a cubic foot of ice of specific gravity ·92. Find whether the combination will float or sink in water.

11. A man who weighs 12 stones can just float in water with a certain quantity of cork (specific gravity ·24) attached to him. Find the weight of the cork, assuming that the man's specific gravity is 1·12.

12. A cylindrical piece of basalt (specific gravity 3), whose height is 6 inches and the radius of whose base is 7 inches, is floating in mercury, (specific gravity 13·6). What force will be required to submerge it?

13. A certain body just floats in water. On placing it in sulphuric acid of specific gravity 1·85, it requires the addition of 42·5 grammes to immerse it. Find its volume.

14. A body, which weighs 12 lbs., floats in a liquid with 1/3 of its volume above the surface. What is the least pressure that must be applied to keep the body wholly submerged?

15. A canal is carried over a road by means of a bridge. Will there be

any increase in the pressure on the bridge when a heavily-laden barge is passing over the bridge?

16. A hollow cylinder of length one foot, and internal diameter one inch, is closed by a thin plate whose weight *in water* is $\frac{1}{2}$ oz. The cylinder is then immersed vertically in water with the closed end downwards to a depth of 7 inches, and a liquid is gently poured into the cylinder. If the plate just falls off when the liquid inside is at the same level as the water outside, find the specific gravity of the liquid. ($\pi = 22/7$.)

17. A vessel of water, in which a piece of cork is floating, is placed under the receiver of an air-pump. Does the cork rise or sink in the water when the pump is worked?

18. A piece of wood floats partly immersed in water, and oil is poured on the water until the wood is completely covered. Will there be any increase or decrease in the portion of the wood below the surface of the water?

19. A vessel contains water and mercury (specific gravity 13·6), and a body floats half in the mercury and half in the water. Find its specific gravity.

20. A piece of metal of specific gravity 7·5 floats in mercury of specific gravity 13·6. Water is poured on the mercury till it just covers the piece of metal. What fraction of the metal is now immersed in the mercury?

21. A solid of specific gravity 2·5 weighs apparently 20 grammes when immersed in water. Find its apparent weight when immersed in acid of specific gravity 1·2.

What will be its apparent weight when *half* immersed in water?

22. A cubic inch of iron weighs 1/4 of a pound. If a cubic foot of iron is suspended by a string and half immersed in alcohol (specific gravity ·8), what will be the tension of the string?

23. If the specific gravity of iron be 7·6, what will be the apparent weight of a cwt. of iron when weighed in water, and how many pounds of wood of specific gravity ·6 will be required to be attached to the iron so as just to float it?

24. The force required to keep one of two bodies submerged in alcohol, (specific gravity ·8), is the same as that necessary to support another body immersed in it. The volumes of the bodies are 11 and 9 cubic centimetres respectively, and the weight of the first is 12·1 grammes. Find the specific gravities of the bodies.

25. A 7-lb. iron ball (specific gravity 7·6) is suspended from one end of an equal-armed lever, and immersed in water. What weight must be hung from the other end of the lever in order to counterpoise it?

26. A body whose volume is 5 cubic feet, and specific gravity 1·2, hangs by a string suspended in water, which completely covers it. What force exerted along the string is required to support the body?

27. A body, whose specific gravity is 1·4, and volume 3 cubic feet, is placed in a vessel in which there is enough water to cover it. What

pressure does the body produce on the points of the bottom of the vessel at which it is supported?

28. A body, weighing 12 lbs., and having a specific gravity of ·75, is fastened by a thread to the bottom of a vessel. When water is poured in, so that the body is completely covered, to what tension is the thread exposed?

29. The specific gravity of iron is 7·6. A piece of iron, which weighs 50 lbs., is supported by a string with 3/4 of its volume immersed in water. What is the tension of the string?

30. A piece of wood (sp. gr. ·85) is weighed in air with platinum weights (sp. gr. 21·5), and appears to weigh 234 grammes. What is the true weight in vacuo, taking the density of air to be 1/770 of the density of water?

B.

31. A ship, weighing 1000 tons, goes from fresh water to salt water. If the area of the section of the ship at the water-line be 15,000 square feet, and her sides vertical where they cut the water, find how much she will rise, taking the specific gravity of salt water to be 1·026.

32. A steamer, in going from salt to fresh water, is observed to sink 2 inches, but after burning 50 tons of coal to rise 1 inch. Prove that the steamer's original displacement was 6500 tons, assuming that the steamer is wall-sided near the water-line, and that the density of salt water is to the density of fresh water as 65 to 64.

33. A steamer, loading 30 tons to the inch in the neighbourhood of the water-line in a fresh-water dock, is found, after a 10-days' voyage, burning 60 tons of coal per day, to have risen 2 feet in salt water at the end of the voyage. Prove that the original displacement of the steamer was 5720 tons, taking a cubic foot of fresh water to weigh 62·5 lbs., and a cubic foot of salt water 64 lbs.

34. A hollow cylindrical vessel floats upright in a liquid with 1/6 of its length immersed. When 20 lbs. of water are poured into it, 5/6 of its length are immersed, and the liquids inside and outside stand at the same level. Find the weight of the cylinder, and the specific gravity of the liquid, neglecting the thickness of the material of which the vessel is composed.

35. A box, made of six equal square boards whose thickness is small compared with the length of a side, floats in water. Compare the depth of the part immersed when the box is water-tight with what it would be if the box were not water-tight.

Apply your result to the case in which the edge of the cube is 20 inches, the thickness of the boards is 1/4 of an inch, and the specific gravity of the wood is 5/6.

36. A cube floats in a liquid of uniform density with one edge below the surface, and three in the surface. Show that the specific gravity of the liquid is six times that of the cube,

37. A triangular lamina ABC floats vertically in a liquid with the angle B in the surface, and the angle A not immersed. Prove that AC is vertical, and compare the densities of the liquid and the lamina.

38. A cylinder of wood of specific gravity s, height h, and area of cross section A, is floated with its axis vertical in a cylindrical tub, containing liquid whose specific gravity is s'. If A' denote the area of the cross section of the tub, show that the level of the liquid in the tub will rise $hsA/s'A'$, when the cylinder of wood is introduced.

Find, in terms of the given quantities, the least depth of liquid in the tub which would just float the cylinder.

39. A sphere of density ρ, and radius r, rests in a cylindrical vessel, radius $2r$, whose base is horizontal. What volume of liquid of density 2ρ must be poured into the vessel to make the sphere float?

40. A cube whose edge is a, and specific gravity s, floats with four edges vertical in a liquid whose specific gravity is ns. Another liquid, whose specific gravity is ms, where $m < n$, which does not mix with the former, is poured on the former liquid so as to cover the cube. Compare the volumes of the liquids displaced by the cube.

41. Two liquids which do not mix are placed in the same vessel. The density of the lower is ρ, and that of the upper is $n\rho$. A cylinder, whose density is $m\rho$, floats in them with its axis vertical, and is completely submerged. Find the condition that it may be half in the upper and half in the lower liquid.

42. A right circular cone, of uniform density ρ, floats just immersed with its axis vertical and vertex downwards in a vessel containing two liquids which do not mix, and whose densities are $m\rho$ and $n\rho$ respectively, where $m > n$. Show that the plane separating the two liquids divides the axis of the cone in the ratio

$$\sqrt[3]{(m-n)/(1-n)} - 1 : 1.$$

43. The external and internal surfaces of a wooden bowl are concentric hemispheres of radii a and b respectively. It is filled with water, and then covered by a circular lid made of the same wood of radius a and thickness $a-b$. Being now set floating in water, the bowl is found to be immersed with the lower face of the lid in the surface. Find the specific gravity of the wood.

44. A spherical shell, made of material of specific gravity s, floats half immersed in water. Find the ratio of the internal radius to the external radius.

45. A uniform rod AB can turn freely about a hinge at A which is immersed at a depth $AB/2\sqrt{3}$ below the surface of a liquid whose density is three times that of the rod. Find the position in which the rod will remain at rest.

46. If, in the preceding Example, the depth of A were AB/2, show that when the rod is in equilibrium a weight equal to 1/4 of the weight of the rod, if attached to B, will depress the rod 15°.

CHAPTER X.—STABILITY OF FLOATING BODIES.

94. Statement of the General Problem.

If a body, which is floating in a position of equilibrium either wholly or partly immersed in a liquid, is slightly displaced from that position, and then left free to move, it may behave in one of three ways:—(i) It may, after some oscillations, come to rest in the position from which it was disturbed; (ii) it may move still farther from that position; (iii) it may remain at rest in the displaced position.

In the first case the body is said to be floating in **stable** equilibrium, in the second case in **unstable** equilibrium, and in the third case in **neutral** equilibrium.

To determine the nature of the equilibrium of a floating body we must consider what will be the effect of the forces acting on the body in a displaced position. These forces are the weight of the body, acting vertically downwards through the centre of gravity of the body, and the resultant pressure of the liquid on the body, a force acting vertically upwards through the centre of gravity of the liquid displaced. The equilibrium of the body is stable or unstable according as these forces tend to bring the body back to its original position, or to cause it to move still farther from that position; and the equilibrium is neutral if these forces are in equilibrium.

In all the figures of this chapter the centre of gravity of the floating body is marked G; and the centre of gravity of the liquid displaced by the body is marked H. This latter point is called the **centre of buoyancy**.

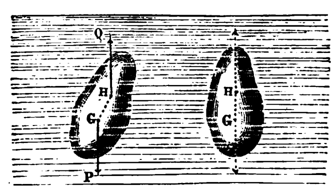
95. Body Floating Wholly Immersed.

First suppose that the body is of the same uniform density as the liquid. Then the body will float wholly immersed at any depth, and every position will be a position of equilibrium. For in this case the weight of the body is equal to the weight of liquid displaced, and the points H and G coincide. Hence in every position of the body, as long as it is totally immersed, the upward pressure of the liquid will be equal and opposite

to the weight of the body. The body will therefore remain at rest in every position.

Next suppose that the weight of the body is equal to the weight of its own volume of the liquid, and that G, the centre of gravity of the body, does not coincide with H, the centre of gravity of the liquid which it displaces. The body will float, wholly immersed, at any depth, but only in a position in which H and G, which are fixed points in the body, are in the same vertical line (Art. 88). Hence there are two positions of equilibrium, one in which H is vertically below G, the other in which G is vertically below H.

Of these two positions it is easy to see that the latter is *stable* and the former *unstable*. For if the body is disturbed from either of these two positions, the forces P and Q (say), the weights of the body and of the liquid displaced, acting at G and H respectively, will form a system of two unlike equal parallel forces, that is, will form a couple. And it is evident from the figure that this couple always tends to turn the body into the position in which G is vertically below H. This position, therefore, is the only position of stable equilibrium, that is, the only position in which the body would remain permanently at rest.



When a body, weight W, which floats wholly immersed in a liquid, is displaced into a position in which GH is inclined to the vertical at an angle θ, the moment of the couple tending to turn the body back to the position of stable equilibrium

$= W \times$ perpendicular distance between the vertical lines through G and H,

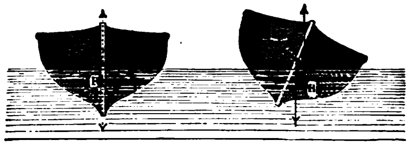
$= W \, . \, \mathrm{GH} \, . \sin \theta.$

96. Body Floating Partly Immersed.

When a body floats partly immersed in a liquid, the line HG is vertical in a position of equilibrium. It is required to determine the nature of the equilibrium when the body is displaced from this position into a position in which HG is in-

clined to the vertical. This case derives great importance from its application to the stability of ships when rolling or pitching in a sea-way. We shall see that in this case the equilibrium may be stable although G is above H.

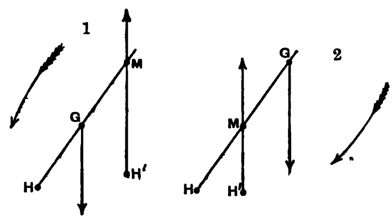
We shall simplify the problem by supposing that the volume of the liquid displaced is not altered by the displacement of the body, and that the centre of buoyancy in the displaced position of the body lies in the vertical plane through the original and displaced positions of the centre of gravity of the body. These conditions will be fulfilled in all cases which we shall consider.



Let W denote the weight of the body. When the body is displaced so that HG, which is a line *fixed in the body*, is inclined to the vertical, the centre of buoyancy changes its position from H to some point which we shall denote by H′. The position of this point will evidently depend on the form of that part of the body which is now under the surface of the liquid. In the displaced position the forces acting on the body are (1) W, the weight of the body, acting vertically downwards through G, and (2) the resultant pressure of the liquid on the body, acting vertically upwards through H′. Since by supposition the volume of the liquid displaced remains unchanged, the resultant pressure of the liquid will still be equal to W. Thus the body in the displaced position is acted on by two equal parallel and unlike forces. These forces therefore form a couple, whose force is W, and whose moment is $W \times$ perpendicular distance between the vertical lines through G and H′. The equilibrium is stable or unstable according as this couple tends to turn the body back to the original position of equilibrium or to turn it farther from that position.

The two cases of stable and unstable equilibrium are illustrated in the accompanying figures 1 and 2 respectively. In figure 1 the couple, formed by W acting vertically downwards at G and W acting vertically upwards at H′, tends to restore the body to

the position of equilibrium. In figure 2 this couple tends to make the body move farther from that position. The direction in which the couple tends to turn the body is indicated in each figure by an arrow-head.



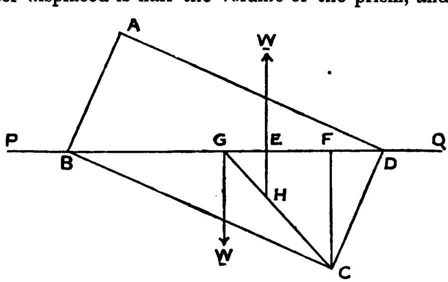
Let the vertical through H′ meet the line HG in the point M, and let the angle GMH′ be denoted by θ, so that θ is the displacement in angle of the body, then, *in the case shown in figure 1*, the moment of the couple tending to turn the body back (called *the righting moment*) is

$$= W \times \text{perpendicular from G upon H'M},$$
$$= W \,.\, \text{GM} \,.\, \sin\theta.$$

Ex. 1.—A right prism has a rectangular cross section ABCD; its specific gravity is ·5; it floats in water with its axis horizontal, and with the shorter sides AB and CD of its cross section vertical; it is displaced so that the diagonal BD is on the surface of the water; find the moment of the couple tending to bring it back to its original position of equilibrium.

The figure represents a section of the prism made by a plane through its centre of gravity, G, perpendicular to its axis. The point G is the middle point of the diagonal BD, which is in the surface PQ of the water.

The volume of the water displaced is half the volume of the prism, and therefore, since the specific gravity of the prism is 1/2, the weight of water displaced is equal to the weight of the prism. Hence in the position under consideration the prism is acted on by two forces, viz., (i) its own weight, W say, acting vertically downwards through G, (ii) the resultant pressure of the water on the part of the prism immersed, a force equal to W, acting vertically upwards through H, the centre of gravity of the water displaced. It is evident that H is the centre of gravity of the triangle BCD.



These two forces form a couple tending to restore the body to the posi-

tion in which AB and CD are vertical. If the vertical through H meets BD in the point E, the moment of this couple is $W \cdot \text{GE}$.

Let $\text{AD} = a$, and $\text{AB} = b$, then GE can be expressed in terms of a and b. Let CF be the vertical through C; then

$$\begin{aligned}\text{GE} &= \tfrac{1}{3}\,\text{GF (since GH} = \tfrac{1}{3}\,\text{GC)} \\ &= \tfrac{1}{3}\,(\text{GD} - \text{DF}), = \tfrac{1}{3}\,(\text{GD} - \text{DC}^2/\text{DB}), \\ &= \tfrac{1}{3}\,(\tfrac{1}{2}\sqrt{a^2+b^2} - b^2/\sqrt{a^2+b^2}), \\ &= (a^2 - b^2)\,/\,6\sqrt{a^2+b^2}.\end{aligned}$$

Hence the moment of the couple tending to bring the prism back to the original position of equilibrium is $W(a^2 - b^2)/6\sqrt{a^2+b^2}$.

Ex. 2.—A uniform rod AB, length a, density ρ, is free to turn about a smooth hinge at A, which is immersed to a depth b in a liquid whose density is $n\rho$, where $n > 1$. If w denotes the weight of unit length of the rod, find the algebraical sum of the moments of the forces acting on the rod in the position in which it is inclined to the vertical at an angle θ, and find the position in which the rod will remain at rest.

[There is a position of equilibrium in which the rod is vertical with the end B below the hinge. This position is evidently unstable, for if the rod is slightly disturbed from this position, the upward pressure of the liquid will tend to make the rod move farther away from the equilibrium position. For, since $n > 1$, this upward force is greater than the weight of the rod, and therefore, since the upward pressure of the liquid and the weight of the rod act in opposite directions in the same straight line, the resultant of these two forces will be an upward force tending to raise the rod into a position in which it is partly immersed.]

Take the figure of page 161 to represent the rod in the position in which it is partly immersed and inclined to the vertical at an angle θ.

Then $\theta = \text{angle DAC}$, $\text{AB} = a$, $\text{AC} = \text{AD}/\cos\theta, = b/\cos\theta$. The forces acting on the rod are (i) the weight of the rod, W say, acting vertically downwards through G, the middle point of AB; (ii) the resultant pressure of the liquid, R say, acting vertically upwards through H, the middle point of AC; and (iii) the reaction of the hinge on the rod.

The force (iii) has no moment about A, and the algebraical sum of the moments of the forces W and R about A

$$\begin{aligned}&= (W \cdot \text{AG} - R \cdot \text{AH}) \sin\theta, \\ &= \tfrac{1}{2}\,(Wa - Rb/\cos\theta) \sin\theta.\end{aligned}$$

On putting $W = wa$, and $R = nw \cdot \text{AC}, = nwb/\cos\theta$, this expression becomes

$$\tfrac{1}{2}\,w\,(a^2 - nb^2/\cos^2\theta) \sin\theta.$$

If this expression is positive, the forces tend to increase θ; if it is zero, the rod will be in equilibrium; and if it is negative, the forces will tend to diminish θ.

Let a be the value of θ which satisfies the equation

$$a^2 - nb^2/\cos^2\theta = 0,$$

that is, let
$$a^2 \cos^2 \alpha = nb^2,$$
or
$$\cos \alpha = b\sqrt{n}/a,$$

then α is the inclination of the rod to the vertical in the position of equilibrium.

It can be shown that this position is one of stable equilibrium. For if the rod is slightly *depressed* from this position into a position in which the inclination to the vertical is θ, so that $\theta > \alpha$, then $\cos\theta < \cos\alpha$ (by trigonometry), and therefore $nb^2/\cos^2\theta > nb^2/\cos^2\alpha$, that is, $nb^2/\cos^2\theta > a^2$. Hence $a^2 - nb^2/\cos^2\theta$ is negative, and the forces in the displaced position tend to raise the rod to the equilibrium position. Also if the rod be slightly *raised* from the position of equilibrium, so that $\theta < \alpha$, then $\cos\theta > \cos\alpha$ (by trigonometry), and $nb^2/\cos^2\theta < nb^2/\cos^2\alpha$, that is, $nb^2/\cos^2\theta < a^2$. Hence $a^2 - nb^2/\cos^2\theta$ is positive, and the forces in the displaced position tend to depress the rod to the equilibrium position. Hence the position in which the rod is inclined to the vertical at the angle α, where $\cos\alpha = b\sqrt{n}/a$, is the position of stable equilibrium.

By putting $b = a/2$, $n = 3$, we get the problem of Ex. 3. Art. 93. In this case α is the angle whose cosine is $\sqrt{3}/2$, that is, the angle 30°. This agrees with the solution of the problem obtained in the example referred to.

[We have supposed that $b\sqrt{n}$ is less than a. If $b\sqrt{n}$ is equal to a, or greater than a, the only position of stable equilibrium is the position in which the rod is vertical, and the centre of gravity of the rod above the fixed point. For in the position in which the rod is inclined to the vertical at an angle θ, the algebraical sum of the moments of the forces *tending to diminish* θ is

$$\tfrac{1}{2} w \,(nb^2/\cos^2\theta - a^2)\sin\theta.$$

It is easy to see that this is positive for all values of θ. For nb^2 is equal to, or greater than, a^2, and $\cos^2\theta$ is always less than 1; therefore $nb^2/\cos^2\theta$ is always greater than a^2. Hence in every position of the rod, the forces acting on the rod tend to turn the rod into the position in which it is vertical.]

Ex. 3.—A uniform rod, length a, specific gravity s, is fastened by one end to a point at a height b above the surface of water into which the other end dips. The rod can turn freely about the point. Under what circumstances will it remain at rest in a vertical position?

Let sw denote the weight of unit length of the rod.

When the rod is inclined to the vertical at an angle θ, the moment of the forces *tending to diminish* θ is

$$\tfrac{1}{2} w \,[sa^2 - (a^2 - b^2\sec^2\theta)]\sin\theta.$$

If this is positive, the forces will tend to turn the rod into the position in

which it is vertical. Hence the rod will remain at rest in the vertical position if, when θ is a small angle,

$$sa^2 > a^2 - b^2 \sec^2 \theta.$$

Now when θ is a small angle, $\sec \theta = 1$, very nearly. Hence the rod will remain at rest in the vertical position if

$$sa^2 > a^2 - b^2.$$

This condition will be satisfied if $s > 1$; for then $sa^2 > a^2$, and therefore $> a^2 - b^2$.

The condition will also be satisfied if $s = 1$.

If $s < 1$, the condition will be satisfied only when

$$b^2 = \text{or} > a^2 - a^2 s$$

that is, when $$b = \text{or} > a\sqrt{1-s}.$$

Hence we conclude that the rod will remain at rest in the vertical position if the specific gravity of the rod is equal to, or greater than, unity. If, however, the specific gravity of the rod is less than unity, the rod will remain at rest in the vertical position only when b is equal to, or greater than, $a\sqrt{1-s}$.

EXAMPLES XII.

(*The Answers are given on page* 336.)

1. A solid sphere, whose centre is C and centre of gravity G, and whose weight, W, is equal to the weight of its own volume of water, is totally immersed in water with the line CG inclined to the vertical at an angle θ. Express, in terms of W, CG, and θ, the moment of the couple tending to turn the sphere into the position of stable equilibrium.

If $W = 100$ lbwt., and CG $= 3$ feet, give numerical results for the following cases:—(i) $\theta = 30°$, (ii) $\theta = 45°$, (iii) $\theta = 60°$, (iv) $\theta = 90°$.

2. A body is composed of two cylinders of wood, of equal cross section, joined end to end with their axes in the same straight line. The lengths of the cylinders are 8 feet and 6 feet, and the specific gravities are 3/4 and 4/3 respectively. Show that the combination will float totally immersed in water.

If the body is displaced from the position of stable equilibrium through an angle of 30°, find the moment of the couple tending to bring it back to its position of equilibrium, the weight of the body being 1000 lbs.

3. A body is made up of a hemisphere, whose radius is 1 foot and specific gravity 5/4, surmounted by a cone, whose axis is $2\frac{1}{2}$ feet and specific gravity 4/5. The radius of the hemisphere is equal to the radius of the base of the cone, and the axes of the hemisphere and the cone are in the same straight line. Show that this combination will float totally immersed in water.

If the body is displaced from its position of stable equilibrium through

an angle of 60°, find the moment of the couple tending to bring it back to the position of stable equilibrium, the weight of the body being 4000 lbs.

4. A uniform rod AB, whose length is 9 feet and specific gravity 4/9, is hinged at the end A to a point fixed at a depth 6 feet below the surface of water. Show that the rod will float upright, and that the force at the hinge will be half the weight of the rod.

5. A thin uniform rod, weighing 1000 grains, whose specific gravity is 2/3, has a weight, which is to be treated as a particle, fastened to one end. Show that if the weight is sufficient to make the rod float vertically in water, it must not be less than 225 grains.

6. A uniform rod, length a, is free to turn round one end which is attached to a point fixed at a height b above the surface of a liquid into which the rod dips. If the specific gravity of the liquid is n times that of the rod, $n>1$, show that if $b=a\sqrt{1-1/n}$, the rod will remain at rest in the vertical position (see Ex. 3, page 171), and the reaction of the fixed point on the rod will be equal to

$$W\sqrt{n-1}\{\sqrt{n}-\sqrt{n-1}\},$$

where W is the weight of the rod.

If $W=1000$ grains, and $n=3$, the reaction of the fixed point is 449 grains very nearly.

7. A uniform rod of weight W is partly immersed in a liquid whose specific gravity is n times that of the rod, and is partly supported by a string attached to the upper end. The string passes over a smooth pulley, and carries a weight W', which is less than W. Show that the rod will remain at rest in the vertical position if W' is not less than

$$W\sqrt{n-1}\{\sqrt{n}-\sqrt{n-1}\}.$$

97. The Metacentre.

Def. *If a body is floating partly immersed in a liquid, and if it receives a small displacement, the point of intersection of the vertical line through the centre of buoyancy in the displaced position with the line in the body which was the vertical line through the centre of gravity of the body in the position of equilibrium, is called the metacentre.*

In this definition the word *displacement* must be taken to mean a displacement such as is considered in the preceding Article, that is, a displacement such that the volume of liquid displaced is not altered, and that the centre of buoyancy in the displaced position lies in the vertical plane through the original and displaced positions of the centre of gravity of the body.

Thus in the figures of page 169 the vertical through H′, the

position of the centre of buoyancy in the displaced position of the body, meets the line HG, which was the vertical line through the centre of gravity in the position of equilibrium, in the point M, so that the position of M when the angle HMH′ is *very small* is, by definition, the position of the metacentre.

It will appear from examination of the figures 1 and 2, that for *small* displacements the equilibrium will be stable or unstable according as M is above or below G. For if M is above G, the couple formed by the weight of the body and the resultant pressure of the liquid will tend to restore the body to the position of equilibrium; and if M is below G, this couple will tend to increase the inclination of GH to the vertical. If M coincides with G, the forces acting on the body in the displaced position will be a system of two equal and opposite forces, and the body will remain at rest in this position. The original position will therefore be one of neutral equilibrium.

Hence, for *small* displacements the equilibrium of a floating body will be *stable*, *unstable*, or *neutral* according as the metacentre is *above*, *below*, or *coincides with* the centre of gravity of the body.

98. **Transverse and Longitudinal Metacentres in a Ship.**

In the case of a ship there are two displacements in which the vertical through H meets the line HG, one being a transverse displacement, and the other a longitudinal displacement. A transverse displacement is one produced by rolling, and a longitudinal displacement is one produced by pitching. Thus there is a metacentre for transverse displacements and a metacentre for longitudinal displacements. We shall call the metacentre for transverse displacements *the* metacentre of the ship; and the height of this point above the centre of gravity we shall call the *metacentric height* of the ship.

In the definition of the metacentre we have supposed that the displacement is very small. In general, the position of the point M will depend on the angle of displacement, but in ships the form of the part under water is such that the position of the point M is nearly the same for all displacements.

In the case in which M is above G, the moment of the couple tending to restore the ship to its position of equilibrium is, by Art. 96, $W \cdot GM \sin \theta$, where W represents the weight of the ship (called the displacement of the ship), and θ is the angle through which the ship has been displaced. Every-

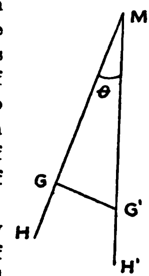
thing else being the same, this moment varies as GM, the height of the metacentre above the centre of gravity. It follows that stability is gained by lowering the centre of gravity of the ship, and so increasing the metacentric height. Hence the advantage of the use of ballast in ships.*

If M is below G, the ship is said to have a *negative metacentric height*, and it would be unstable. This is the case with some ships, which are stable only when laden with cargo or ballast. In commercial phraseology, such ships "cannot shift without ballast."

99. Inclining a Ship to determine its Metacentric Height.

The height of the metacentre of a ship above its centre of gravity may be found by the following experiment, which is known in naval architecture as "the inclining experiment":—

A weight of several tons, which is already on board the ship, whose total displacement is known, is moved across the deck through a measured distance. The ship, in consequence of the change of position of its centre of gravity due to this change in the position of the weight, floats in a position of equilibrium in which it is inclined at a small angle to the upright. The inclination of the ship to the vertical in this position is determined by the measured deflection of a pendulum. From these data the metacentric height of the ship can be determined.



In the figure G and H are the positions in the body of the centre of gravity of the body and of the centre of buoyancy respectively before the weight is moved. G′ and H′ are the positions of the same two points after the weight is moved, and M is the metacentre.

In the original position of equilibrium HGM is vertical, and in the inclined position of equilibrium H′G′M is vertical.

Let W be the total displacement of the ship, w the weight moved, a the distance through which it is moved, and θ the small angle which the ship in the inclined position makes with the vertical, that is, the angle HMH′.

The weight w is moved through the distance a perpendicular to GM, and therefore (Ex. 3, page 56) the centre of gravity of the ship is in consequence moved through the distance aw/W in a parallel direction. Hence GG′ is perpendicular to MG, and equal to aw/W.

Now

$$GG' = GM \tan \theta;$$

therefore

$$GM \tan \theta = aw/W,$$

from which

$$GM = aw/W \tan \theta,$$

a formula which gives the metacentric height in terms of known quantities.

* The metacentric height is usually a small number of feet. Cleopatra's Needle was brought to England in a circular pontoon whose metacentric height was 6 inches.

If θ is a small angle, we may write for $\tan\theta$ the radian measure of θ.* Thus we have approximately

$$GM = aw/W\theta.$$

The angle θ may be measured directly, or it may be found, very approximately, by measuring the distance through which the end of a pendulum of known length moves due to the inclination of the ship to the vertical. Thus, if the end of a pendulum of length l moves through a distance b, the radian measure of θ is approximately equal to b/l. Hence the formula for GM may be written in the form

$$GM = alw/Wb.$$

Ex.—A ship of 7000 tons displacement has a weight of 30 tons moved 50 feet across the deck, and a deviation of 12 inches in 15 feet is produced. What is the metacentric height?

Here

$$W=7000,\ w=30,\ a=50,\ l=15,\ b=1.$$

Hence

$$GM = 50\times 30\times 15/7000\times 1, = 45/14,$$
$$= 3\tfrac{3}{14} \text{ feet.}$$

100. Position of the Metacentre in the case of a Sphere Floating Partly Immersed.

If that part of the floating body which is under the surface of the liquid is in the form of a portion of a spherical surface, the metacentre will coincide with the centre of the sphere.

For the pressure on every element of the spherical surface, being perpendicular to the surface, acts along the radius, and therefore the resultant pressure of the liquid is, in every position of the floating body, a vertical force passing through the centre of the sphere.

Hence in the position of equilibrium the centre of gravity lies in the vertical line through the centre of the sphere. Therefore the centre of the sphere is the point in which the vertical through H′ meets the line HG (see figures, page 169), that is, the centre of the sphere is, by definition, the metacentre.

From this we deduce two results:—

(i) When a sphere, whose centre of gravity coincides with its centre of figure, floats partly immersed in a liquid, the

* See the Author's *Trigonometry*, page 157.

equilibrium is *neutral*. For in this case the centre of gravity coincides with the metacentre.

(ii) When a sphere, whose centre of gravity does not coincide with its centre of figure, floats partly immersed in a liquid, the only position in which the sphere will remain at rest is that in which the centre of gravity of the sphere is vertically below its centre of figure. For the condition for a position of stable equilibrium is that the centre of gravity be vertically below the metacentre.

This result applies to the cases of a loaded sphere, a sphere containing a cavity, and a body in the form of a portion of a uniform sphere.

101. Formula for HM.

If a body, which is floating partly immersed in a liquid, receives a displacement such as is considered in Art. 97, the position of the metacentre for that displacement is determined by the following formula, which we assume without proof:*—

$$\mathrm{HM} = Ak^2/V.$$

This formula gives the height of M, the metacentre, above H, the centre of buoyancy in the position of equilibrium, in terms of

V, the volume of the liquid displaced by the body;
A, the area of the horizontal section of the body at the surface at the liquid;

and k, the radius of gyration of this area about the axis in this plane about which the body is displaced.

Now the equilibrium is stable or unstable according as M is above or below G, that is, according as HM is greater or less than HG. (We suppose H to be below G.)

Hence the equilibrium is

stable if $Ak^2/V > HG$,
and **unstable** if $Ak^2/V < HG$.

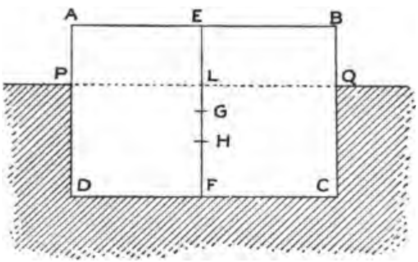
Thus we can determine the character of the equilibrium of a

* For a proof of this formula the advanced student may consult the article on "Hydrodynamics" in the *Encyclopædia Britannica*, vol. xii.

floating body for a given displacement when the values of V, A, k and the positions of H and G, are known.

Ex. 1.—Show that a uniform cylinder of wood, whose specific gravity is 2/3, will float in water in stable equilibrium with its axis vertical if the radius of its cross section is greater than two-thirds of its height.

Let ABCD represent a section of the cylinder made by a plane through the axis in the position in which the cylinder is floating with its axis vertical. It is required to prove that if the radius is greater than two-thirds of the height of the cylinder, this position will be stable.



Let the length of EF, the axis, be denoted by h, and the length of AE, the radius, by r. The centre of gravity of the cylinder, G, is the middle point of EF, and the centre of buoyancy, H, is the middle point of LF, the part of the axis which is immersed. We shall apply the formula for HM to prove that, for a displacement about an axis through L perpendicular to the plane of the diagram, the equilibrium will be stable if r is greater than $2h/3$.

Since the specific gravity of the cylinder is 2/3, it follows that

$$\mathrm{LF} = 2h/3. \qquad \text{(Art. 90.)}$$

Hence $$\mathrm{HG} = \mathrm{FG} - \mathrm{FH}, \ = h/2 - h/3, \ = h/6.$$

Also $$\mathrm{HM} = Ak^2/V,$$

where A is the area of the cross section of the cylinder, k the radius of gyration of this area about an axis in its plane passing through its centre, and V is the volume of water displaced.

Now $$V = A \times \mathrm{LF}, \ = 2Ah/3,$$
and $$k^2 = r^2/4. \qquad \text{(Ex. (ii) page 77.)}$$

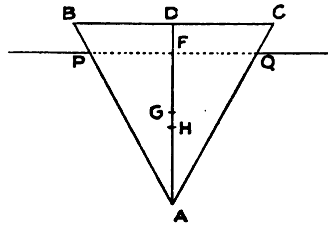
Hence $$\mathrm{HM} = (Ar^2/4) \,/\, (2Ah/3), \ = 3r^2/8h.$$

The metacentre will be above the centre of gravity if

$$\mathrm{HM} > \mathrm{HG};$$
that is, if $$3r^2/8h > h/6;$$
that is, if $$r^2 > 4h^2/9;$$
that is, if $$r > 2h/3.$$

Hence the equilibrium will be stable if $r > 2h/3$.

Ex. 2.—Find the position of the metacentre of a cone, floating in water with its axis vertical and vertex downwards, in terms of its dimensions and specific gravity, and find the condition for stability.



Let ABC be a section of the cone made by a plane through the axis AD. Let h represent the length of the axis AD, r the radius BD of the base, h' the length of the part AF of the axis immersed, r' the radius PF of the water-line section, and s the specific gravity of the cone.

Then $\quad \mathrm{HM} = Ak^2/V, \; = Ak^2/\tfrac{1}{3}Ah', \; = 3k^2/h'$
$\quad = 3r'^2/4h', \quad (\text{since } k^2 = r'^2/4).$

Now $\quad r'/h' = r/h \quad$ (by similar triangles);

and $\quad r'^2 h' = r^2 hs \quad$ (Art. 90);

therefore $\quad h'^3 = h^3 s,$

from which $\quad h' = h\sqrt[3]{s}.$

Hence $\quad \mathrm{HM} = 3r'^2/4h', \; = 3r^2h'/4h^2,$
$\quad = 3r^2\sqrt[3]{s}/4h,$

which determines the height of the metacentre about the centre of buoyancy.

Also $\quad \mathrm{HG} = \mathrm{AG} - \mathrm{AH}, \; = 3(h - h')/4,$
$\quad = 3h(1 - \sqrt[3]{s})/4.$

The equilibrium will be stable if

$$\mathrm{HM} > \mathrm{HG};$$

that is, if $\quad 3r^2\sqrt[3]{s}/4h > 3h(1 - \sqrt[3]{s})/4;$

that is, if $\quad r^2/h^2 > (1 - \sqrt[3]{s})/\sqrt[3]{s}.$

As a numerical example, suppose that $s = {\cdot}729$, then $\sqrt[3]{s} = {\cdot}9$, and the equilibrium will be stable if r^2/h^2 is greater than $(1 - {\cdot}9)/{\cdot}9$, that is, greater than 1/9, or if r/h is greater than 1/3. But r/h is the tangent of BAD, half the vertical angle of the cone. Hence for this value of s the equilibrium will be stable if the tangent of half the vertical angle of the cone is greater than 1/3.

EXAMPLES XIII.

(*The Answers are given on page* 336.)

1. A ship of 4000 tons displacement, when fully laden with coals, has a metacentric height of $2\frac{1}{2}$ feet. Suppose 100 tons of coal to be shifted so that its centre of gravity moves 18 feet transversely, what would be the angle of heel of the vessel, if she were upright before the coal shifted?

[Given tan 10° = ·1763, tan 11° = ·1944.]

2. The displacement of a vessel is 400 tons, and the transverse metacentre is $5\frac{3}{4}$ feet above the centre of buoyancy. If a weight of 12 tons, already on board, is moved 8 feet across the deck, find the inclination of the vessel to the upright, the centre of gravity of the vessel being 3 feet above the centre of buoyancy.

[Given $\tan 5° = \cdot 087$ nearly.]

3. If a cube, whose specific gravity is $\cdot 5$, is placed in water with four edges vertical, show that its position is one of unstable equilibrium.

4. A cube of wood floats in water. Prove that it cannot rest with a face horizontal if the density of the wood lies between $\cdot 21$ and $\cdot 79$ times that of water.

5. Show that a prism on a square base will float in water with its axis vertical if a side of the base exceeds $\sqrt{6s(1-s)}$ times the height, where s is the specific gravity of the prism.

6. Show that a cylinder, whose specific gravity is $\cdot 8$, will float in water with its axis vertical if the ratio of the radius to the length of the axis is greater than the ratio of $2\sqrt{2}$ to 5.

7. A cylinder floats in water with its axis vertical. Show that if s is the specific gravity of the cylinder, r the radius of its base, and h its height, the length of HM is $r^2/4hs$, and of HG is $h(1-s)/2$.

Hence show that when $s = 9/11$, the condition for stable equilibrium is that the ratio of the radius of the base to the length of the axis shall be greater than 6/11.

8. Show that a conical wooden buoy, whose specific gravity is 8/27, cannot float in water with its axis vertical and vertex downwards unless its semi-vertical angle exceeds the angle whose tangent is $\cdot 707$.

9. Show that a conical wooden buoy of specific gravity s cannot float in water with its axis vertical and vertex downwards unless its semi-vertical angle exceeds the angle whose tangent is

$$\left\{ \sqrt[3]{1/s} - 1 \right\}^{1/2}$$

10. The thickness of the material of a hollow cylinder is small in comparison with the radius; it is open at both ends; find the greatest height it can have if it is to float with its axis vertical.

11. The thickness of the material of a hollow cylinder with open ends is small in comparison with the radius; a fine ring is fastened round one end of it, whose weight is a ninth part of the weight of the cylinder; the specific gravity of the material of the cylinder is $\cdot 8$. Show that it will float in stable equilibrium with the axis vertical and the ring out of the water if its height is less than $90/\sqrt{1520}$ or $2\frac{4}{13}$ of the radius.

12. A rectangular beam, density σ, floats in a liquid density ρ. The edges are $2a$, $2b$, $2c$, the latter being vertical in the position of equilibrium. Discuss the stability of the equilibrium for a slight displacement in which the edges $2a$ remain horizontal.

CHAPTER XI.—PNEUMATICS. ATMOSPHERIC PRESSURE.

Properties of Gases, Arts. 102 to 105.

102. Liquids and Gases.

There are some properties which are common to the two classes of fluids—liquids and gases. Like a liquid, *a gas has weight,* has no tendency to maintain a definite shape, that is, *has no elasticity of shape,* and after compression due to increase of pressure, returns to its original volume when the original pressure is restored, that is, *has perfect elasticity of volume.*

At a given point in a mass of gas the pressure is the same in all directions, and is measured in the same way as the pressure at a point in a liquid. Also the pressure will vary from point to point of a mass of gas at rest according to the laws of Art. 57.

On the other hand, there are certain properties exhibited by gases, which are not possessed by liquids. In the first place, *a gas tends to expand indefinitely,* so that it can be kept only in a closed space. The bounding surface which encloses a gas may be a rigid surface, *e.g.* the surface of a boiler enclosing a mass of steam, or the surface of a liquid, *e.g.* the film of liquid in a soap bubble filled with hydrogen. In the case of the air in a diving-bell sunk in water, we have an example of a gas bounded partly by rigid surfaces and partly by the surface of a liquid. In all cases, a gas exerts pressure on the surfaces which enclose it, and the surfaces react with equal and opposite forces on the gas.

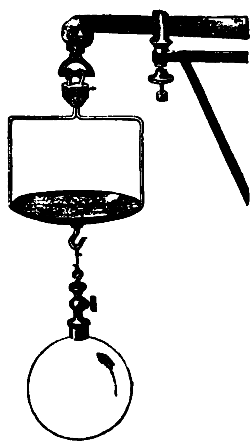
Again, *gases are compressible,* while liquids offer great resistance to compression. Thus, while the volume of a given mass of liquid is always nearly the same, the volume of a given mass of a gas depends on the pressure to which it is subjected.

Pneumatics treats of the properties of gases, and in particular of the relations between the volume, the pressure, and the temperature of a given mass of gas. Some of the properties of atmospheric air, which may be taken as the type of gases, are

illustrated by the simple experiments described in the following two Articles:—

103. Weight of Air.

That air has weight may be demonstrated by the experiment of weighing a glass globe, *first*, when the globe has been exhausted of air by means of the air-pump, and *secondly*, after the air has been re-admitted to the globe. It is found that the weight of the globe when full of air exceeds its weight when exhausted, the difference between the two weights representing the weight of air which fills the globe. The same method may be employed to determine the weight of a known volume of any other gas.



In carrying out the experiment a globe, furnished with a stop-cock, is taken, and, when exhausted of air, is suspended from one of the scale-pans of a balance. The weight which counterpoises the globe is the weight of the globe when empty. The stop-cock is now opened, and the globe becomes filled with air. The weight which now counterpoises the globe is the weight of the globe when full of air. The difference between the two weights is the weight of air which fills the globe.

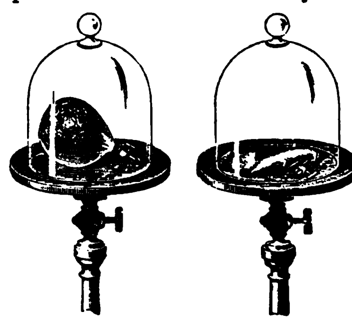
The student must notice that in *each* of the weighings the weight which counterpoises the globe is the *apparent* weight of the globe, and is, by the principle of Archimedes, less than the true weight by the weight of the volume of air displaced by the globe. The *difference* between the apparent weights is equal to the *difference* between the true weights; and since the difference between the true weights is equal to the weight of air in the globe, it follows that the difference between the apparent weights is also equal to the weight of air in the globe.

It is found by experiment that the weight of a given volume of air at ordinary temperatures and pressures is equal to about 1/773 of the weight of the same volume of water.

104. Expansibility of Air.

The expansibility of air may be illustrated by placing a closed bladder, containing some air, under the receiver of an air-pump, and working the pump. As the air is withdrawn from the receiver, the pressure of the air on the outer surface of the bladder is reduced, the air inside the bladder expands, and the bladder becomes inflated. On re-admitting the air to the receiver, the bladder shrinks to its original condition.

This experiment shows that air tends to expand, and that the volume of a given mass of air depends on the pressure to which the air is subjected. Also the return of the bladder to its original condition when the original pressure is restored, illustrates the statement that the elasticity of volume of a gas is perfect.


Expanding Bladder.

105. Pressure of a Gas in a Closed Vessel.

Since a gas has weight, it follows from the principles of Hydrostatics that in a mass of gas in equilibrium the intensity of pressure will vary from point to point. It was proved in Art. 57 that the difference of intensities of pressure at two points in any fluid is equal to the weight of a column of the fluid, whose section is unit area, and whose height is equal to the difference of levels of the two points. In the case of a gas enclosed in a vessel the difference of level of any two points will always be small, and therefore, since the weight of gases is small, we may without sensible error neglect the differences of pressure due to weight. Thus since the density of air at ordinary pressure is only about 1/773 of the density of water, it follows that the difference of the intensities of pressure at two points in air, whose levels differ by a foot, is equal to the pressure due to a height of 1/773 of a foot of water, that is, is equal to the very small pressure of about $57/10^5$ lbwt. per square inch.

In the case, therefore, of a gas enclosed in a vessel, we shall always neglect differences of pressure due to difference of level. It follows that the pressure of the gas will be the same at all points, and will be measured by the force exerted by the gas on unit area of the surface of the vessel. This is called the **pressure of the gas in the vessel.** Thus the pressure of steam in a boiler is measured by the force, usually expressed in pounds weight, exerted by the steam on a square inch of the surface of the boiler.

Methods of determining by experiment the pressure of a gas will be explained in the next chapter.

Pressure of the Atmosphere, Arts. 106 to 115.

106. The Atmosphere.

The atmosphere surrounding the earth is an example of a gas which is not enclosed in a vessel. In this case the gas is prevented from expanding indefinitely by the force of gravity which attracts it to the surface of the earth.

The atmosphere exerts on the surface of all bodies a pressure whose intensity at points near the sea-level is about 15 lbwt. per square inch. This pressure is due to the weight of the column of air, of sectional area one square inch, extending from the surface of the earth to the limits of the atmosphere. It follows from the principles of Hydrostatics that this pressure must become less as we rise above the sea-level (Art. 57), the difference of pressures at two points at different levels representing the weight of a column of air of unit sectional area and height equal to the difference of levels of the two points. This is in accordance with the results of observations taken in balloon ascents and in the ascents of mountains. Thus at the top of Mont Blanc the atmospheric pressure has been found to be only 3/5 of the atmospheric pressure at the sea-level.

107. Experiments illustrating Atmospheric Pressure.

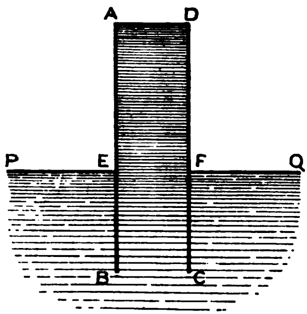
Experiment 1.—Fill a glass with water, close the mouth of the glass with a card, and press the card against the glass with the finger. On inverting the glass, and removing the finger, it will be found that the pressure of the atmosphere acting upwards on the card will support the weight of the card and of the water in the glass.

Experiment 1.

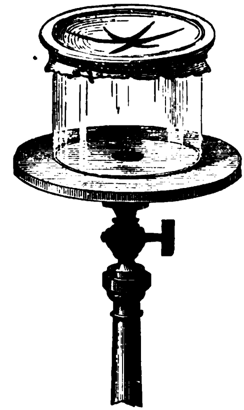
Experiment 2.—A long glass cylinder, ABCD (see figure, page 185), which is open at the end BC but closed at the end AD, is immersed in water. When the cylinder is full of water, it is partly withdrawn from the water into the position shown in the figure, in which the end BC is below the surface, PQ, of the water. It is found that the cylinder remains full of water, so that inside the cylinder there is a column of water whose height above the water outside is equal to the height of the upper end, AD, above PQ, the surface of

the water outside. This column, AEFD, of water is supported by the pressure of the atmosphere on the surface PQ, this pressure being transmitted through the water to the open end BC of the cylinder.

Experiment 3.—Another experiment which shows clearly the pressure of the atmosphere

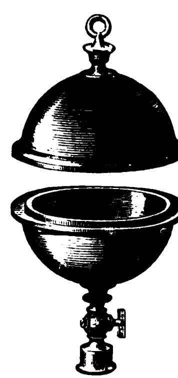


Experiment 2.



Experiment 3.

on the surface of bodies may be performed by means of an air-pump. A piece of some thin elastic material, such as the material of a child's balloon, is tied over one end of an open glass cylinder. The cylinder is then placed with its other end on the plate of an air-pump, and the air within it is gradually withdrawn by working the pump. The pressure of the air inside the cylinder is thus reduced below the pressure of the external air, and after the pump has been worked for some time, the difference between the external and internal pressures becomes so great that the elastic covering bursts.



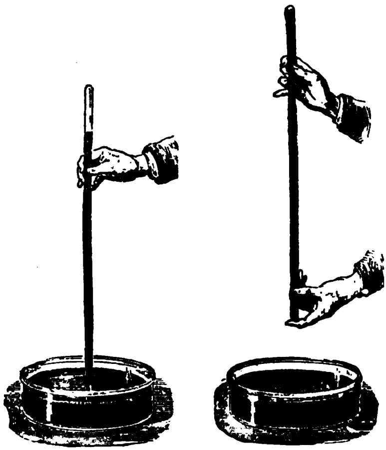
Experiment 4.

Experiment 4.—Two hemispheres, one of which is furnished with a stop-cock, are fitted together, and the air within them is then withdrawn by means of an air-pump. The stop-cock is closed, and it is found that the hemispheres cannot be separated without the application of considerable force. The explanation of this is that the hemispheres are pressed together by the pressure of the external air, and there is no internal pressure. If the stop-cock is now opened, the air will rush into and fill the space enclosed by the hemispheres, and the internal pressure will become equal to the external pressure. It will then be found that the cylinders may be easily separated. This experiment is known as the experiment of the *Magdeburg hemispheres.*

108. Torricelli's Experiment.

The following experiment is known as the Torricellian experiment. It was first performed by Torricelli, an Italian philosopher, in 1643.

A glass tube more than 32 inches long, open at one end and closed at the other end, is held in a vertical position and filled with mercury. The tube is then inverted, the open end being closed with the finger to prevent the escape of the mercury, and is placed in a vertical position in a basin of mercury, with its lower end below the surface of the mercury. On removing the finger from the end of the tube it is found that the mercury subsides from the top of the tube, the surface descending to some point from 28 to 31 inches above the surface of the mercury in the basin. The space above the surface of the mercury in the tube contains no air, and is almost a perfect vacuum. It is often referred to as the *Torricellian vacuum.**



The column of mercury in the tube is supported by the pressure of the atmosphere, acting on the surface of the mercury in the cistern. This pressure is transmitted through the mercury to the open end of the tube, and supports the weight of the column of mercury in the tube.

That the weight of the column of mercury in the tube is supported by the atmospheric pressure may be demonstrated by placing the tube and cistern of mercury under the receiver of an air-pump, and working the pump. As the air is withdrawn from the receiver, the pressure on the surface of mercury

* The space is not a perfect vacuum, as it contains vapour of mercury. (Art. 141.)

in the cistern is reduced, and the mercury falls in the tube. On readmitting the air to the receiver, the mercury rises again, and stands at its former level.

The supposition that it is the atmospheric pressure that supports the column of mercury in the tube is also in accordance with the observed diminution in the length of the column, as the experiment is performed at points higher and higher above the sea-level.

109. The Mercurial Barometer.

Def.—*A barometer is an instrument for measuring the pressure of the atmosphere.*

The account of the Torricellian experiment, given in the preceding article, may be taken as a description of the method of constructing a mercurial barometer in its simplest form. This form of the barometer consists essentially of a tube, filled with mercury, inverted in a vertical position with its open end under the surface of mercury in an open cistern. In the construction of barometers for the purpose of determining with great accuracy the atmospheric pressure, certain precautions must be taken in order to expel bubbles of air from the mercury and to thoroughly clean the sides of the barometer tube.

The height to which the column of mercury stands in the tube is called the **height of the barometer.** This height varies from time to time at the same place, and from place to place at the same time.

110. Intensity of Atmospheric Pressure.

The intensity of atmospheric pressure may be calculated when the height of the mercurial barometer and the weight of a given volume of mercury are known.

Let p denote the intensity of atmospheric pressure, h the height of the mercurial barometer, and w the weight of unit volume of mercury. Also, let a denote the area of the section of the tube.

The volume of the column of mercury in the tube is ah, and its weight is wah. This weight is supported by the pressure of intensity p, exerted over the area of the section, a, of the tube. Hence

$$pa = wah,$$

from which

$$p = wh \text{..............} (1)$$

This formula gives the intensity of atmospheric pressure in *gravitational* units. If h is the height of the barometer in inches, and w the weight in pounds of a cubic inch of mercury, then wh is the atmospheric pressure in pounds weight per square inch. If h is the height in centimetres, and w the weight in grammes of a cubic centimetre of mercury, then wh is the atmospheric pressure in grammes weight per square centimetre.

If ρ is the density, or mass of unit volume, of mercury, and g the acceleration of gravity at the place of observation, then w is equal to $g\rho$ absolute units of force. Hence the intensity of atmospheric pressure is given in *absolute* units by the formula

$$p = g\rho h \text{..............} (2)$$

In formula (2), if h is expressed in inches, ρ in pounds per cubic inch, and g in foot-second units, then p will be expressed in poundals per square inch. If h is expressed in centimetres, ρ in grammes per cubic centimetre, and g in centimetre-second units, then p will be in dynes per square centimetre.

It follows from formula (2) that, if g and ρ are taken to be constant, the pressure of the atmosphere will be measured by h, the height of the barometer. Since g varies slightly from point to point on the earth's surface, and since ρ varies with the temperature, it is not strictly true that the same height of the barometer represents at all places on the earth's surface, and at all temperatures, the same atmospheric pressure.

111. Corrections to be applied to a Reading of the Barometer. The height of the barometer is read off on a scale attached to the barometric tube. To this reading certain corrections must be applied

before it can be taken to be an exact measure of the atmospheric pressure. The most important of these corrections are—

1. The correction for capillarity.
2. The correction for temperature.
3. The correction for variation in the value of g.

Correction for capillarity.—In all tubes of small bore containing mercury, the column is subject to a slight depression due to capillarity.* The amount of this depression varies with the diameter of the tube, being greater the smaller the diameter. Tables have been constructed which give the amounts of depression for tubes of different diameters. For a tube of 1/5 of an inch in diameter, the depression is about 3/50 of an inch.† A reading of the height of the barometer is corrected for capillarity by adding to the reading the depression given in the table for tubes of the diameter equal to that of the barometric tube.

Corrections for temperature and for variation in the value of g.—If h is the height of the column of mercury, corrected for capillarity, at a place where the intensity of gravity is g, and if ρ is the density of mercury at the temperature of the air when the reading is taken, then the atmospheric pressure is $g\rho h$ absolute units of force per unit of area. Now g varies slightly from place to place on the earth's surface, and ρ changes slightly due to changes of temperature. It follows that, in comparing the atmospheric pressures at different places, under different conditions of temperature, by means of the heights of the barometers, we must take into account the fact that the same length of column does not represent at all places, and at all temperatures, the same pressure. The height of the barometer must first be *reduced to standard temperature at the standard place*, that is, we must find what would be the length at some standard temperature of the column whose *weight* at some standard place would be equal to the weight of the barometric column at the place of observation.

The standard temperature is generally taken to be the freezing temperature, and the height of the barometer is reduced to that temperature. If t° C. is the observed temperature, h the observed height of the barometer, and if h' denote the length of column which the mass of mercury in the tube would occupy at freezing temperature, then it is found by experiment that

$$h' = h\,\{1 - 18t/10^5\}. \quad \ldots\ldots\ldots\ldots \quad (1)$$

Next let g be the acceleration of gravity at the place of observation, and g' the acceleration of gravity at the standard place, usually taken to be sea-level at latitude 45°. Also, let h'' denote the length of column at freezing temperature, whose weight at the latter place would be equal to the weight

* See Chapter xviii.

† See Deschanel's *Natural Philosophy*, Part I., page 150, for a table of capillary depressions in barometric tubes.

of the length h' at the place of observation. Then, since the densities and weights of the columns are equal,

$$g'h''=gh',$$

from which
$$h''=gh'/g. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Substituting in this formula, the value of h' in terms of h from equation (1), we get finally

$$h''=gh\,(1-18t/10^5)/g'. \quad \ldots\ldots\ldots \quad (3)$$

The value of h'', which is obtained from equation (3), is the height of the barometer reduced to zero temperature at the standard place. The atmospheric pressures at different places, and under different conditions of temperature, are proportional to the heights of the barometer reduced to standard temperature at the standard place.

Ex. 1.—What is the intensity of pressure of the atmosphere in pounds weight per square inch when the height of the mercurial barometer is 30 inches?

The specific gravity of mercury is 13·6, and a cubic foot of water weighs 62·5 lbs.

The pressure per square inch is equal to the weight of a column of mercury, one square inch in section, and 30 inches high.

Hence the pressure is equal to the weight of 30 cubic inches of mercury, and this

$$=\frac{30}{12\times12\times12}\times62\cdot5\times13\cdot6,$$
$$=14\cdot8 \text{ lbwt.}$$

Ex. 2.—Find the intensity of atmospheric pressure in dynes per square centimetre at a place where g is 981 cm./sec.^{2}, when the height of the barometer is 760 millimetres, taking the specific gravity of mercury to be 13·6.

The intensity of pressure in dynes per square centimetre is equal to the weight in dynes of a column of mercury whose sectional area is one square centimetre and height 760 millimetres. The volume of this column is 76 cubic centimetres; its mass is $76\times13\cdot6$ grammes; and its weight at the given place is $76\times13\cdot6\times981$ dynes. Hence the intensity of atmospheric pressure is, in dynes per square centimetre,

$$=76\times13\cdot6\times981$$
$$=10^6\times1\cdot014 \text{ nearly.}$$

Hence the atmospheric pressure is approximately equal to the pressure of one million dynes per square centimetre.

Ex. 3.—If a denotes the sectional area, taken to be uniform, of the tube in the cistern barometer, and A the area of the surface of mercury in the cistern, show that when the surface of the mercury in the tube rises through an additional height h, the surface of mercury in the cistern will fall

through a distance ah/A, and that the mercurial column will be lengthened by the amount $h\ (A+a)/A$.

If k denote the amount by which the surface of mercury in the cistern falls when the surface of mercury in the tube rises through a height h, then the increase of volume of the mercury in the tube is ah, and the diminution in the volume of the mercury in the cistern is Ak; and these are equal.

Hence $$Ak=ah,$$
from which $$k=ah/A.$$

Also, the increase in the length of the mercurial column in the tube

$$\begin{aligned} &=h+k=h+ah/A \\ &=h\ (A+a)/A. \end{aligned}$$

112. The Siphon Barometer.

The siphon barometer consists of a bent tube with its two limbs, which are straight and parallel, of unequal lengths. The longer limb, which is as long as the tube of the ordinary barometer, is closed, and the shorter limb is open to the atmosphere. The tube is fixed with its limbs vertical, and contains mercury, the space above the mercury in the longer limb being a Torricellian vacuum.

The mercury stands higher in the closed branch than in the open branch. The pressure of the atmosphere, acting on the surface of mercury in the open branch, supports the weight of the column of mercury, whose height is equal to the *difference* of the levels in the two limbs.

Siphon Barometer.

Aneroid Barometer.

113. The Aneroid Barometer.

In the aneroid barometer no fluid is used, the variations of atmospheric pressure being measured by the rise and fall of the top of a closed metallic box. The top of the box, which is partially exhausted of air, is made of thin material, and rises and falls according as the atmospheric pressure

diminishes or increases. By a combination of levers any motion of the top of this box A is communicated to, and causes rotation in, a drum F. A pointer H turns with the drum, so that any rise or fall of the top of the box actuates the pointer. The amount of a movement of the pointer is indicated on a dial plate placed below the pointer. The instrument is graduated by the maker by comparing its indications of a mercurial barometer at different times.

114. Heights of Barometers filled with different Liquids.

Barometers may be constructed by employing water, glycerine, or other liquid, instead of mercury. At the same place and time the heights of barometers filled with different liquids will be different.

It is easy to prove that the heights of two barometers filled with different liquids are inversely proportional to the specific gravities of the liquids.

Let h_1 and h_2 denote the heights of two barometers filled with liquids of specific gravities s_1 and s_2 respectively, and let w denote the weight of unit volume of water. Then the weights of unit volumes of the liquids are ws_1 and ws_2 respectively. Hence by Article 110 the intensity of atmospheric pressure is equal to wh_1s_1, and also to wh_2s_2.

Hence

$$wh_1s_1 = wh_2s_2,$$

from which

$$h_1s_1 = h_2s_2,$$

or

$$h_1 : h_2 = s_2 : s_1.$$

Hence *the height of the barometric column is inversely proportional to the specific gravity of the liquid employed.* Since the specific gravity of mercury is much higher than that of any other liquid, it follows that the length of the column is much shorter in the mercurial barometer than in a barometer filled with water, glycerine, or other liquid.

Thus the **height of the water barometer** would be equal to $13{\cdot}6 \times$ height of mercurial barometer. Taking the height of the mercurial barometer at 30 inches, we find that the height of the water barometer would be

$$= 13{\cdot}6 \times 30 \text{ inches}, = 34 \text{ feet}.$$

115. Pressure of an Atmosphere.

In stating the results of experiments in which very great pressures are employed, as in recent experiments on the liquefaction of gases, it is usual to take, as the unit of pressure, the pressure of the atmosphere, which is referred to as the **pressure of an atmosphere.** As the atmospheric pressure varies from place to place, and from time to time, the pressure of an atmosphere is not an invariable unit. We may, however, take 30 inches, or 76 centimetres, as the standard height of the barometer, so that the expression "*pressure of an atmosphere*" may be taken to be a pressure of 15 lbwt. per square inch.

It has been proposed to use, as the pressure of an atmosphere, the pressure of a million dynes per square centimetre. This pressure does not differ much from the pressure of 76 centimetres of mercury—see Ex. 2, page 190, and is, of course, absolutely invariable.

Ex. 1.—Taking the specific gravity of mercury at 13·6, what would be the height of an oil barometer, when the mercury barometer stands at 29 inches, the specific gravity of the oil being ·845?

Let x be the height in inches of the oil barometer, and let w represent the weight of a cubic inch of water. Then the intensity of atmospheric pressure in lbwt. per sq. inch is $x \times w \times {\cdot}845$, and also equal to $29 \times w \times 13{\cdot}6$. Equating these expressions, we get

$$x \times w \times {\cdot}845 = 29 \times w \times 13{\cdot}6;$$

from which

$$x = 29 \times 13{\cdot}6/{\cdot}845 \text{ inches}, \\ = 38{\cdot}9 \text{ feet.}$$

Ex. 2.—A vertical tube is full of water, and its lower end stands in water, the surface of which is $8\frac{1}{2}$ feet below the top of the tube. What is the pressure of the water against the top of the tube in pounds weight per square inch, taking the height of the water barometer to be $33\frac{1}{2}$ feet?

In the figure on page 185 let ABCD be the tube, the open end BC being below the surface PQ of the water, so that the height of AE is $8\frac{1}{2}$ feet. Consider the equilibrium of the column EADF of water. This column is at rest under the action of three vertical forces: (i) its weight, (ii) the upward pressure of the atmosphere across the section EF, this pressure being transmitted from the surface PQ through the liquid; (iii) the reaction of the top of the tube AD, a force acting downwards equal and opposite to the pressure exerted on the top of the tube by the liquid inside the tube. The

sum of the downward forces (i) and (iii) must be equal to the upward force (ii).

The force (ii) is equal to the weight of a column of water, of the same sectional area as the tube, and of a height equal to $33\frac{1}{3}$ feet. The force (i) is the weight of $8\frac{1}{3}$ feet of this column. Hence the force (iii) is equal to the weight of $(33\frac{1}{3} - 8\frac{1}{3}), = 25$, feet of this column. Therefore, the pressure of the water inside the tube on the top of the tube is an upward force equal to the weight of a column of water, of the same sectional area of the tube, and of a height equal to 25 feet. Hence the pressure per square inch will be the weight of a column of water, one square inch in section, and 25 feet high.

Taking the weight of a cubic foot of water to be 62·5 lbwt., we find the pressure on the top of the tube to be

$$= 25 \times 12 \times 62{\cdot}5/1728,$$
$$= 10{\cdot}85 \text{ lbwt. per square inch.}$$

Ex. 3.—A tube, partly filled with mercury, is inverted in a vertical position with its open end below the surface of mercury in a basin. If the mercury stands to a height of 20 inches, what is the pressure of air above the mercury in the tube, the height of the barometer being 30 inches?

The air at the top of the tube exerts a downward force on the mercury in the tube, and this force, together with the weight of the column of mercury, is equal to the upward pressure of the atmosphere exerted across a section of the tube.

Let p denote the intensity of pressure of the air above the mercury, this pressure being expressed in terms of a height of mercury (Art. 66), then

$$p + 20 = 30,$$
giving
$$p = 10.$$

Hence the intensity of pressure of the air above the column of mercury is that due to a depth of 10 inches of mercury.

Since a pressure of 30 inches of mercury is equivalent to a pressure of 14·8 lbwt. per square inch (Ex. 1, page 190), it follows that the intensity of pressure of the air above the mercury is $14{\cdot}8/3, = 4{\cdot}94$ lbwt. per square inch.

EXAMPLES XIV.

[A cubic foot of water weighs 1000 oz.]

(The Answers are given on page 336.)

1. Supposing that a barometer is set up in a position not exactly vertical, would this affect the reading?

2. Would the reading of a barometer be affected by inequalities in the sectional area of the tube?

3. The upper ends of two long vertical glass tubes are connected together

and to the receiver of an air-pump. The lower end of one dips into a beaker of water, that of the other into a beaker of sulphate of copper. On working the air-pump, the liquids rise in the two tubes, but to different heights. Explain the cause of this.

4. Taking the atmospheric pressure at 15 lbwt. per square inch, find the pressure on the bottom of a tank, filled with water, the depth of the tank being 2 feet, and the area of the bottom of the tank, 2 square yards.

5. Find the pressure in lbwt. per square inch at a point 20 feet below the surface of water, freely exposed to the atmospheric pressure of 15 lbwt. per square inch.

6. Determine the pressure in lbwt. per square inch, and also in atmospheres, at a depth of 30 feet of water exposed to atmospheric pressure, taking the atmospheric pressure at 15 lbwt. per square inch.

7. Determine the greatest depth in fathoms at which a submarine diver can work in sea water, supposing he can bear a pressure of 5 atmospheres, taking an atmosphere to be a pressure of 15 lbs. per square inch, and a cubic foot of sea water to weigh 64 lbs.

8. A tube filled with water is inverted with its open end in water, no air having got in, and the top of the tube is 20 feet above the surface of the external water. If the water barometer stands at 34 feet, what is the pressure in lbs. per square foot at a point on the inside of the top of the tube?

What would be the consequence of making a small hole through the top of the tube?

9. A vessel, exhausted of air, is closed by a valve in the form of a square whose side is one foot long. The valve can turn round one edge. If the atmospheric pressure is 15 lbwt. per square inch, what is the force which must be applied at the centre of the valve to open it?

Also, what is the force which must be applied at the middle point of the edge opposite to that about which the valve can turn, in order to open it?

10. If the valve in the preceding example is in the form of a circle, 6 inches in diameter, which can turn about a hinge at a point A in its circumference, what is the force which must be applied at the point B, the other extremity of the diameter through A, in order to open the valve, the atmospheric pressure being 15 lbwt. per square inch?

11. A square valve, which can turn round an edge, closes an exhausted receiver. If the area of the valve is 100 square inches, what is the force that must be applied to its middle point in order to open it, when the mercurial barometer stands at 30 inches? [Take the specific gravity of mercury to be 13·6.]

12. If the hemispheres (in Experiment 4, page 185) are completely exhausted, find the force required to separate them, taking the diameter of the hemispheres to be 6 inches, and the atmospheric pressure to be 15 lbwt. per square inch.

13. A barometer tube, whose internal diameter is $\frac{3}{4}$ of an inch, weighs,

when empty, 4 lbs. If the mercury in the tube stands at a height of 30 inches, find the whole vertical pressure of the tube on its supports, taking the specific gravity of mercury to be 13·6.

14. The heights of the barometer, corrected for capillarity, at two places, A and B, are 30·25 and 25·95 inches respectively, the temperatures of the air at the times of the observations being 20° C. and 5° C. respectively. The accelerations of gravity at A and B are 32·2 and 32·08 respectively. Express correct to 3 places of decimals, the ratio of atmospheric pressure at A to the atmospheric pressure at B at the times of the observations.

15. Show that the whole mass of the atmosphere is the same as that of an ocean of pure water about 33 feet deep.

Taking the radius of the earth to be 21 million feet, show that the mass of the atmosphere is about 5×10^{15} tons.

Chapter XII.—BOYLE'S LAW.

116. Laws of Gases.

The variations in the volume of a gas due to variations of pressure or temperature are subject to two laws, known as *Boyle's Law* and *Charles' Law* respectively.

Boyle's Law states the relation between the pressure and the volume of a gas, *when the temperature is constant.* The truth of this law for atmospheric air appears to have been discovered independently by two natural philosophers, Boyle in England and Marriotte in France, who lived in the latter part of the 17th century. The question of priority is one on which English writers and continental writers are not agreed. By the latter the law is usually referred to as the Law of Marriotte.

Charles' Law states the relation between the volume and temperature of a gas which expands or contracts *under constant pressure.* This law is sometimes assigned to Dalton, but it appears to have been first discovered by Charles. We shall always speak of it as the Law of Charles.

In this chapter we shall consider Boyle's Law and its applications. It must be understood that all the results we shall arrive at are obtained on the supposition that changes of temperature are not considered.

117. Boyle's Law.

The pressure of a given mass of a gas is inversely proportional to its volume, as long as the temperature is kept constant.

This statement of Boyle's Law may be expressed in symbols. Let p denote the pressure of the gas when the volume is v, and p' the pressure when the volume is changed to v'. Then Boyle's Law is expressed by the proportion—

$$p : p' = v' : v;$$

from which $$pv = p'v'.$$

Hence the product of the volume and pressure is the same in the two cases. It follows that for the same mass of gas, the product of the volume and the pressure is the same in all cases. Hence we say that Boyle's Law leads to the equation

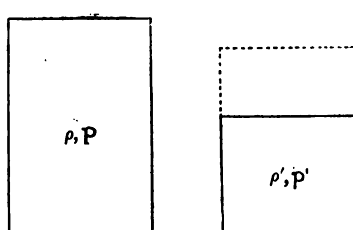
$$pv = C,$$

where C is a quantity which is constant for the same mass of the same gas. C will be different for different gases and for different masses of the same gas.

The following deduction from Boyle's Law is very important—

When the temperature is constant, the pressures of two masses, equal or unequal, of the same gas are proportional to their densities.

Let ρ and p denote the density and pressure respectively of a mass of gas, and let ρ' and p' have the same meanings for another mass of the *same* gas. Let the volume of the latter mass of gas be changed so that the density is changed from ρ' to ρ, and let x and y denote the original and altered volumes. [The volumes of the two masses of gas may be represented diagrammatically, as in the accompanying figure.]



The two masses will then be masses of the same gas of the same density ρ, and they will therefore have the same pressure p. By Boyle's Law—

$$py = p'x \dots\dots\dots (1)$$

But since there is no change in the mass of the gas whose volume is changed, we have, by Art. 23,

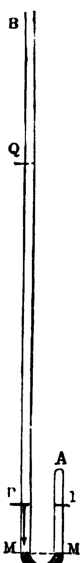
$$\rho y = \rho' x \ldots\ldots\ldots (2)$$

From equations (1) and (2) we get

$$p/\rho = p'/\rho',$$

which was to be proved.

Thus, for example, if there is a mass of air in a closed vessel, and if, by pushing in a piston, we reduce the volume of the vessel by a half, the density of the air in the vessel will be doubled, and therefore by Boyle's Law the pressure will also be doubled. Again, if, without altering the volume of the vessel, we introduce into the vessel a mass of air equal to the mass already in the vessel, the density will be doubled, and the pressure will also be doubled.



118. Experimental verification of Boyle's Law.

The Law of Boyle may be verified for atmospheric air by the following experiments. In Experiment 1 the verification is for pressures *greater* than the atmospheric pressure, and in Experiment 2 for pressures *less* than atmospheric pressure.

Experiment 1.—In this experiment a bent glass tube AB of uniform bore is used, the branches of the tube being straight and parallel, but of unequal lengths. The tube is fixed with its branches vertical. Both branches of the tube being open, a small quantity of mercury is poured into the tube so as to fill the curved portion MM of the tube, and the shorter end A is then sealed up. We have then imprisoned in MA a quantity of air at the pressure of the atmosphere as indicated by the height of the barometer at the time and place of the experiment.

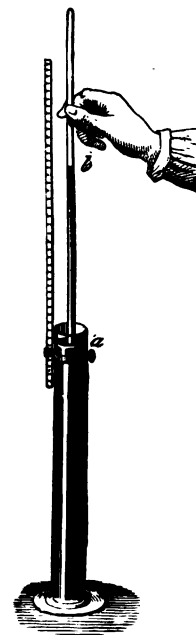
Mercury is again poured into the tube at the open end B until the level of the mercury in the closed branch AM of the tube rises to the point N, midway between A and M, so that the volume of the imprisoned air is now half its original volume. Suppose that Q is the point at which the mercury stands in the open branch when the mercury has risen to N in the closed branch. Let P be the point on the open branch on a level with N.

Since N and P are on the same level in the same liquid, the pressure at N is equal to the pressure at P. Now the pressure at P is equal to the pressure of the column of mercury PQ, together with the atmospheric pres-

sure on the open surface Q. Hence the pressure at N, that is, the pressure of the air confined in AN, is equal to the sum of the atmospheric pressure and the pressure of the column PQ of mercury.

Now it is found that PQ is equal to the height of the barometer, that is, the pressure at P or at N is double the atmospheric pressure. Hence the air, which under atmospheric pressure occupied the volume AM, under double the atmospheric pressure occupies the volume AN, which is just half the original volume.

Hence by doubling the pressure the volume is halved. In the same way we should find that by increasing the pressure to three times the atmospheric pressure, the air in AM would be compressed to one-third of its original bulk; and so on. Hence the volume varies inversely as the pressure.



The law is thus verified for pressures *greater than atmospheric pressure.*

Experiment 2.—A long, straight glass tube of uniform bore, closed at one end and open at the other, is taken and partly filled with mercury. The tube is inverted and held vertically in a vessel containing mercury, so that it then contains air at a pressure less than atmospheric pressure. The tube is pushed down until the levels of the mercury in the tube and the vessel are the same. In this position the column of air in the tube is at atmospheric pressure. The length of this column is measured. Let l denote the length. The tube is then raised until the length of the column of air is $2l$. It is then found that the mercury has risen in the tube, and stands at a height equal to half the height of the barometer at the time of the experiment. The pressure of the imprisoned air is therefore half the atmospheric pressure, and its volume is double the volume it occupied when its pressure was equal to the atmospheric pressure. It follows that, by allowing the air to expand until its volume is doubled, the pressure is reduced by one-half. On drawing out the tube until the length of the column of air is $3l$, we allow the air to expand to a volume which is three times the volume it occupied at atmospheric pressure. It is then found that the mercury stands inside the tube at a height equal to two-thirds of the height of the barometer. The pressure of air is, therefore, equal to one-third of the atmospheric pressure, so that when the volume of air is increased in the ratio of 3 to 1, the pressure is reduced to one-third of the original pressure. In a similar way the law may be verified for greater volumes.

In this way the law is verified for pressures *less than atmospheric pressure.*

All experiments for verifying Boyle's Law must be carried out slowly. For when the volume of a mass of gas is increased or diminished, a change in the temperature of the gas is produced. Compression of a gas is always accompanied by a rise, and expansion by a fall, of temperature. In performing experiments, therefore, on the compression and expansion of gases in which the temperature is to be kept constant, time must be allowed for the gas, after a compression or an expansion, to return to the temperature of the atmosphere.

119. Departures from Boyle's Law.

Modern experiments have shown that, for changes of pressure which are not very great, the changes in volume in the case of all gases are very approximately in accordance with the law of Boyle, but that there is no gas which obeys this law exactly when the pressures are very great.

It has also been shown that some gases are more compressible than others. For example, it has been found that carbonic acid gas suffers greater compression than common air when subjected to great pressure. If equal volumes of air and of carbonic acid gas are measured off at atmospheric pressure, and each subjected to a pressure of 25 atmospheres, the volume of the carbonic acid gas will be only about 4/5 of the volume of the air.

120. Faulty Barometer.

If some air gets into the tube of a barometer, it will rise in the tube and occupy the space above the mercury. This air will exert pressure downwards on the column EB of mercury, and the mercury will therefore stand lower than in a perfect barometer.

By an application of Boyle's Law we can obtain a formula for the correction to be applied to a reading of a faulty barometer.

Let c represent the length of the tube of a faulty barometer above the surface ABCD of the mercury in the cistern, and let h be the height of the barometer when the true height of the barometer is H. It is required to determine the true height of the barometer when the reading of the faulty barometer is h'.

We assume that the area of the section of the tube of the faulty barometer is uniform, so that the volume of the air

above the mercury is proportional to the length of the tube which it occupies. When the faulty barometer stands at a height h, the length of the column of air in the tube is $c-h$, and the pressure of this air is measured by $H-h$. When the reading of the faulty barometer is h', the air occupies a length $c-h'$ of the tube. If x denotes what is then the true height of the barometer, the pressure of air in the tube is measured by $x-h'$, and this is the correction to be applied to the reading h'.

Thus the pressures of the air in the tube are measured by $H-h$ and $x-h'$ when the volumes are measured by $c-h$ and $c-h'$ respectively. Hence by Boyle's Law—

$$(x-h')\ (c-h')=(H-h)\ (c-h),$$

from which $$x-h'=(H-h)\ (c-h)/(c-h').$$

This gives a formula for the correction, $x-h'$, to be applied to the reading h'. Also, x, the true height, is given by the formula—

$$x=h'+(H-h)\ (c-h)/(c-h').$$

Ex. 1.—A vessel of 3 cubic feet capacity, containing air at 2 atmospheres' pressure, is put into communication with a vessel of 18 cubic feet capacity, containing air at $\frac{1}{4}$ of the atmospheric pressure. What is now the pressure of air in the two vessels?

The air in the first vessel, if allowed to expand until the pressure is equal to the atmospheric pressure, would occupy 2×3, $=6$ cubic feet. The air in the second vessel, if compressed until the pressure is equal to the atmospheric pressure, would occupy $\frac{1}{4}\times 18$, $=4\frac{1}{2}$ cubic feet. Hence the air in the two vessels would, at atmospheric pressure, occupy $(6+4\frac{1}{2})$, $=10\frac{1}{2}$ cubic feet. When the two vessels are put into communication this air actually occupies $(3+18)$, $=21$ cubic feet. The quantity of air, therefore, which would occupy $10\frac{1}{2}$ cubic feet at atmospheric pressure, occupies 21 cubic feet. Hence the pressure of the air is

$$10\tfrac{1}{2}/21,\ =\tfrac{1}{2}\times \text{ pressure of an atmosphere.}$$

Ex. 2.—An open vessel contains air at a place where the barometric pressure is 30 inches of mercury. It is carried to the top of a mountain, where the barometric pressure is 20 inches of mercury. Compare the masses of air in the vessel at the two places.

At the higher station the air in the vessel is at a pressure of 20 inches of mercury. Imagine the vessel to be closed and the volume to be reduced until the pressure of air is increased to that of 30 inches of mercury. The reduced volume would, by Boyle's Law, be 2/3 of the original volume. But

the pressure, and therefore also the density, of air in the vessel would now be the same as at the lower station. The densities being the same, the masses will be proportional to the volumes. Hence the mass of air in the vessel at the higher station would be 2/3 of the mass at the lower station.

Ex. 3.—A barometer into which a little air has got into the upper part is found to record 28 inches when the true barometric height is 30 inches. If the volume of the space above the mercury be $7\frac{1}{2}$ cubic inches, what would be the volume of the air within it at atmospheric pressure?

The pressure of the air above the mercury is a pressure of $(30-28)$, $=2$ inches of mercury. At this pressure the volume of the air is $7\frac{1}{2}$ cubic inches. Therefore, if v denote the volume which this air would occupy at the atmospheric pressure of 30 inches, we have by Boyle's Law—

$$30v = 2 \times 7\tfrac{1}{2};$$

from which

$$v = \tfrac{1}{2} \text{ cubic inch.}$$

Hence at atmospheric pressure the air would occupy a volume of $\frac{1}{2}$ cubic inch.

121. Dalton's Law for a Mixture of Gases.

It is a well-known experimental result that when two gases, between which no chemical action takes place, are put into communication, they form a mixture of uniform density. For example, common air is a mixture of oxygen and nitrogen.

The following law regarding the pressure of a mixture of gas was enunciated by **Dalton** as the result of experiment:—

The pressure of a mixture of two or more gases is equal to the sum of the pressures that would be produced by each of the constituents of the mixture if the other constituent or constituents were not present.

Dalton's Law includes the law of Boyle as a special case. For it follows from Dalton's Law that if any number, n say, of equal masses of the *same* gas are introduced into a vessel, the pressure would be n times the pressure exerted by one of the masses when the other masses are not present in the vessel. Now the density of the gas when the n masses are in the vessel would evidently be n times the density when one only of the masses is in the vessel. Hence it follows from Dalton's Law that when the density of a gas is increased n times, the pressure is also increased n times; and this is precisely the law of Boyle.

From Dalton's Law we can obtain a solution of the following problem:—

Two volumes, v and v', of the same gas, or of two gases which do not act chemically on each other, are mixed together, the mixture occupying a volume u. If p and p' denote respectively the pressures before mixture, it is required to find the pressure of the mixture.

Let x and y denote respectively the pressures which the two masses of gas would separately produce when occupying the volume u. Then by Boyle's Law—

$$ux = pv, \text{ and } uy = p'v',$$

from which

$$x = pv/u, \text{ and } y = p'v'/u.$$

But by Dalton's Law the pressure of the mixture is $x + y$.

Hence the pressure of the mixture, P say, is given by

$$\begin{aligned} P &= pv/u + p'v'/u, \\ &= (pv + p'v')/u. \end{aligned}$$

Ex.—If 1000 cubic inches of air under a pressure of 20 lbs. per square inch are mixed with 800 cubic inches of air under a pressure of 15 lbs. per square inch, find the pressure of the mixture if it has a volume of 1500 cubic inches.

Here

$$\begin{aligned} &v = 1000, \ v' = 800, \ u = 1500, \\ &p = 20 \quad p' = 15, \end{aligned}$$

and it is required to find P.

$$\begin{aligned} P &= (pv + p'v')/u, \\ &= (20 \times 1000 + 15 \times 800)/1500 \\ &= 21\tfrac{1}{3} \text{ lbs. per square inch.} \end{aligned}$$

122. Manometers.

A manometer or pressure-gauge is an instrument for measuring the pressure of a gas.

Thus the gauge by means of which the pressure of steam in a boiler is usually measured is a manometer whose action depends on the elasticity of some metal.

Of manometers whose action depends on pneumatic principles there are two classes: (i) open manometers; (ii) closed or compressed-air manometers.

The open mercurial manometer is an instrument in which the pressure of a gas is measured in terms of the height of the column of mercury which would produce this pressure.

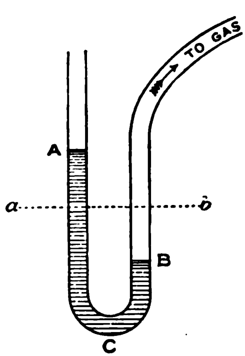
One form of the open manometer consists of a closed vessel

containing some mercury. Through the top of the vessel a vertical tube b passes air-tight, the upper end of the tube being open to the atmosphere, and the lower end being below the surface of mercury in the vessel.

To measure the pressure of a gas by means of this gauge, the gas is admitted at the opening a to the space above the mercury. When the pressure of the gas exceeds the atmospheric pressure the mercury will be forced up the tube b, and the height to which the mercury stands in the tube above the level of the surface of mercury in the vessel is a measure of the *excess* of the pressure of the gas above the atmospheric pressure. Thus if z denote the difference of levels of the mercury in the tube b and in the vessel, and if H denote the height of the mercurial barometer, the *excess* of the pressure of the gas over the pressure of the atmosphere is a pressure of z/H atmospheres, that is, the pressure of the gas is a pressure of $(z/H+1)$ atmospheres.

123. Siphon Manometer.

The siphon manometer is another form of the open pressure-gauge.



In this manometer, of which the accompanying diagram is a sketch, the mercury is contained in a siphon tube of uniform bore, whose branches are straight and parallel, and fixed in a vertical position. The branch CA is open at the top to the atmosphere, and the top of the other branch CB is in communication with the gas.

When the pressure of the gas is equal to the pressure of the atmosphere the mercury stands at the same level, the level ab in the figure, in both branches. When the pressure of the gas exceeds atmospheric pressure the mercury

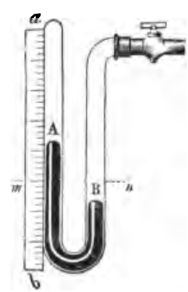
stands higher in the branch CA than in the branch CB, and the *excess* of the pressure of the gas over the pressure of the atmosphere is equal to the pressure of the column of mercury, whose height is the difference of the levels of A and B, the surfaces of the mercury in the two branches. This difference of level is evidently equal to twice the height of A above the level *ab*.

If z denote the height of A above the level *ab*, and H the height of the mercurial barometer, the pressure due to the difference of levels is a pressure of $2z/H$ atmospheres. Hence the pressure of the gas is $(2z/H+1)$ atmospheres.

124. **Compressed-air Manometer.**

If the branch CA of the siphon manometer is closed, the gauge will become a form of the compressed-air manometer. In this instrument there is a quantity of air imprisoned above the surface of the mercury in the branch A. The action of the instrument depends on the diminution of volume of this enclosed air due to increase of pressure.

The gauge is constructed by the maker so that when the pressure of the gas in the branch above B is equal to the atmospheric pressure the mercury stands at the same level, the level *mn* in the figure, in both branches. When the pressure of the gas exceeds the atmospheric pressure the mercury falls in the branch B through a certain distance below the level *mn*, and rises in the branch A through an equal distance. The pressure of the gas *exceeds* the pressure of the air above A by the pressure due to the column of mercury whose height is equal to the difference of levels of A and B, the surfaces of the mercury in the two branches, or, what is the same thing, equal to twice the height of A above the level *mn*.



By an application of Boyle's Law a formula may be obtained for the pressure of the air above A for a given height of A above the level *mn*. By adding to this pressure the pressure due to the column of mercury whose height is equal to the

difference of levels of A and B, we may obtain a formula for the pressure of the gas above B for a given height of A above *mn*.

Let c represent the length of *am*, which is the length of the column of air above A when the pressure of this air is equal to the atmospheric pressure. Let z denote the height of A above the level *mn*, and let p denote *in atmospheres* what is then the pressure of the air. Then the volumes of the air at the pressure 1 and p are measured by c and $c - z$ respectively.

Hence by Boyle's Law—

$$p(c-z)=c,$$

from which

$$p=c/(c-z).$$

Let H denote the height of the mercurial barometer. Then the pressure due to the difference of levels, $2z$, of the mercury in the two branches is $2z/H$ of an atmosphere. Hence, finally, the pressure of the gas above B

$$=p+2z/H,$$
$$=\{c/(c-z)+2z/H\} \text{ atmospheres.}$$

Ex. 1.—If the siphon manometer is to read pressures up to three atmospheres, what quantity of mercury will be required, the area of the section of the tube being a quarter of a square inch?

When the pressure of the gas in the siphon manometer is a pressure of three atmospheres, the difference of levels of A and B (see figure, p. 204) is double the height of the barometer. Hence there must be at least as much mercury in the tube as will occupy a length of the tube equal to twice the height of the barometer. Taking the height of the barometer to be 30 inches, the mercury must fill a length of 60 inches of the tube. Since the section of the tube is a quarter of a square inch, this gives a volume of

$$60 \times \tfrac{1}{4}, =15 \text{ cubic inches.}$$

If the volume of mercury in the tube were less than 15 cubic inches, the pressure of gas would force out the mercury at the open end of the branch CA.

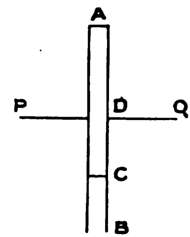
Ex. 2.—If, in the compressed-air manometer, the length of the column of air at atmospheric pressure is 20 inches, what is the pressure of the gas in the branch B (see figure, Art. 124) when the column of air occupies a length of 10 inches of the tube, the height of the barometer being 30 inches?

Since the air has its volume reduced from 20 to 10, its pressure is doubled. Hence its pressure is a pressure of two atmospheres. Also, since the mercury in one branch rises 10 inches, the difference of levels in

the two branches is 20 inches, and the pressure of 20 inches of mercury is equal to 2/3 of an atmosphere.

Hence the pressure of the gas in the branch B is equal to the pressure of $2\frac{2}{3}$ atmospheres.

Ex. 3.—A tube of uniform bore, stopped at one end, and of a length equal to the height of the barometer, is put vertically with its open end downwards into mercury. At what depth must the open end be below the surface of the mercury if the mercury fills one-quarter of the tube?



Let AB be the tube placed vertically with its open end B under PDQ, the surface of the mercury. Let h represent the height of the mercurial barometer, and let z denote DB, the depth of the open end B of the tube below the surface of the mercury.

The mercury rises to C, where $BC = h/4$, and the length of the column of air AC is $3h/4$. The pressure of this air is equal to the pressure in the mercury at the depth of C. This pressure is that due to the height $(h + CD)$ of mercury, that is to the height $(h + z - h/4)$, $= (3h/4 + z)$ of mercury.

Hence the air which occupies a length h of the tube at atmospheric pressure, that is at the pressure measured by h, occupies a length $3h/4$ at the pressure measured by $3h/4 + z$. Therefore, by Boyle's Law—

$$\frac{3h}{4}\left(\frac{3h}{4} + z\right) = h^2,$$

from which $$z = 7h/12.$$

Hence the depth of the open end of the tube below the surface of the mercury is 7/12 of the height of the barometer, that is, 7/12 of the length of the tube.

EXAMPLES XV.

[In all questions the temperature is supposed to be constant.]

(*The Answers are given on page* 337.)

1. A hollow cylinder, fitted with a piston, contains air at a pressure of 15 lbs. per square inch when the piston is 12 inches from the bottom; if more air is forced in till there is three times as much air as at first, and if the piston is allowed to rise 4 inches, what is now the pressure of air per square inch?

2. Assuming that one hundred cubic inches of air weigh 32 grains when the barometer stands at 30 inches, find the weight of air that gets out of a room when the barometer falls from 30 inches to 29·5 inches, the room being 30 feet long, 20 feet wide, and 15 feet high.

3. If 310 cubic centimetres of a gas, measured at a pressure of 750 millimetres of mercury, are subjected to a pressure of 800 millimetres of mercury, what will be the resulting volume of the gas?

4. Find the weight of a cubic foot of air at a pressure of 16 lbs. on the square inch, being given that a cubic foot weighs 360 grains at a pressure of 13 lbs. on the square inch.

5. An open vessel contains one gramme of air at a place where the barometer stands at 750 millimetres. It is then taken to a place where the barometer stands at 700 millimetres. What is now the mass of air in the vessel?

6. A cubic inch of air under a pressure of 15 lbs. per square inch is mixed with two cubic inches of air under a pressure of 45 lbs. per square inch. If the mixture occupies $3\frac{1}{2}$ cubic inches, what is the pressure of the mixture?

7. Two vessels of equal volumes contain air, the pressure in one being 10 lbs. per square inch, and in the other 30 lbs. per square inch. One-third of the air in the first is transferred to the second. What is now the pressure of air in each vessel?

8. A tube 17 inches long, closed at the top, dips vertically into a basin of mercury. The mercury outside rises to the middle point of the tube, and inside it stands 6 inches higher, leaving the rest of the tube filled with air. How far must the tube be drawn out that the contained air may expand so as to fill another half-inch of the tube, the height of the barometer being 30 inches?

9. A quantity of air confined in a graduated tube over mercury measures 20 cubic centimetres when the surface of mercury in the tube stands at 46 centimetres above the surface of mercury in the basin. When the surface of mercury in the tube stands at 10·8 centimetres higher than the mercury in the basin, the volume of the confined air is 9 cubic centimetres. Show what is the height in centimetres of the mercurial barometer at the time.

10. A U-shaped tube of uniform bore is sealed at one end. The tube contains mercury, which rises to the same height on each side of the bend, and which confines in the closed limb a quantity of air occupying a length of 10 inches of the tube. The tube is then put under the receiver of an air-pump, and the pump is worked until the air in the tube occupies a length of 12 inches. Show what is now the pressure of air in the receiver and in the tube respectively, the original pressure of air having been equal to that of 30 inches of mercury.

11. The mercury stands at the same level in the open branch and in the closed branch of a bent tube of uniform section—both branches being vertical—when the air confined in the closed branch is at a pressure of 30 inches of mercury, which is that of the external air. Express in atmospheres the pressure which, acting on the surface of the mercury in the open branch, compresses the confined air to one-third of its original volume, and at the same time maintains a difference of 6 inches in the levels of the two mercurial columns.

12. A U-shaped tube with vertical branches, one closed and the other open, contains mercury, and air above the mercury in the closed branch. When the height of the mercurial barometer is 30 inches the two surfaces

of mercury in the tube are in the same horizontal line, and the enclosed air occupies 29 inches of the length of the tube. The barometer falls, and the enclosed air is observed to occupy 30 inches of the length of the tube. How much has the barometer fallen?

13. A bent tube, having both ends open, is filled with mercury, and fixed so that each end is $6\frac{1}{2}$ inches above the level of the mercury. One end is then closed so as to be air-tight, and the other is gently filled up with mercury, when it is found that the mercury at the closed end rises one inch. Find the height of the mercurial barometer.

14. The height of the top of a uniform barometer-tube is 33 inches above the mercury in the tank, but on account of air in the tube the barometer registers 28·6 inches when the atmospheric pressure is equivalent to 29 inches of mercury. What will be the true height of the barometer when the height registered is 29·93 inches?

15. The mercury in a barometer-tube stands at 30 inches, and has a vacuum of 7 inches above the mercury, the cross section of the tube being 1 square inch in area. If a cubic inch of the external air is let into the tube so as to fill the vacuum space, how many inches will the mercury in the tube fall?

16. A barometer stands at 30 inches, the vacuum above the mercury being perfect. The area of the cross section of the tube is $\frac{1}{4}$ of a square inch. If a quarter of a cubic inch of air is now allowed to get into the barometer, and if the column of mercury falls 4 inches, what was the volume of the original vacuum?

17. The height of the water barometer is $33\frac{1}{3}$ feet. A bubble of air has a volume of one cubic inch at a depth of 100 feet below the surface of pure water. What will be its volume on reaching the surface?

18. A cubic foot of water weighs 1000 oz. A cylindrical test-tube is held in a vertical position, and immersed mouth downwards in water. When the middle point of the tube is at a depth of 32·75 feet it is found that the water has risen half-way up the tube. Find the atmospheric pressure in pounds weight per square inch.

19. A bubble of air, whose volume is one cubic inch, rises from the bottom of the sea 260 metres deep. Find the volume of the bubble on reaching the surface if the barometer reads 760 millimetres. The specific gravities of sea-water and of mercury may be taken to be 1·02 and 13·6 respectively.

20. A uniform tube, 2 feet long, and closed at one end, is at the beginning full of air. The tube is lowered, mouth downwards, into the sea 25 fathoms, and afterwards 150 fathoms. Find the height to which the water will rise in the tube in each case.

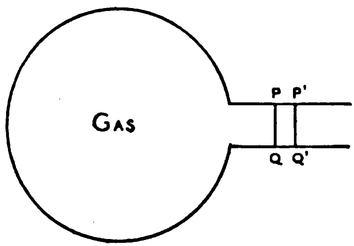
[Five and a half fathoms of sea-water give one atmosphere of pressure.]

Work done by a Gas expanding at Constant Temperature, Arts. 125 to 128.

125. Statement of the problem.

A gas is contained in a closed vessel, and is allowed to expand from one given volume to another given volume; it is required to find the amount of work done by the gas during the expansion, *the temperature being constant.*

We may suppose that the gas is allowed to push out a piston fitted, as in the accompanying sketch, into a cylindrical opening in the side of the vessel. As the piston is pushed out, the volume of the gas increases, and the pressure of the gas does work on the piston.



The problem may be varied by supposing that work is done by an external force in pushing *in* the piston against the pressure of the gas. In this case work will be done *on* the gas by the external force which pushes in the piston.

In a gas expansion is always accompanied by a fall of temperature, and compression by a rise of temperature. If we wish to leave out all consideration of change of temperature, we must suppose that the expansion or compression takes place so slowly that time is allowed, after each small compression or expansion, for the gas to regain its original temperature.

126. Work done by a Gas during a very Small Expansion.

Let v denote the volume of a mass of gas in a vessel, and let p denote its pressure. Imagine that the gas is allowed to expand, at constant temperature, to the volume v', where v' is greater than v by an infinitesimal amount. Then the amount of work done by the gas during the expansion is

$$p\,(v'-v).$$

We know from Boyle's Law that the pressure of a gas diminishes as the volume increases, and therefore in this case the pressure of the gas will diminish as the volume increases from v to v'. But since the change of volume is very small, the change of pressure will be very small, and may be neglected in the calculation of the work done during the expansion.

Let PQ (see figure, preceding article) be the position of the piston when the volume of the gas is v, and P′Q′ the position when the volume is increased to v', so that the volume of PQQ′P′ is equal to $v' - v$. Let A denote the number of units of area in the section of the cylinder in which the piston moves.

The whole force exerted by the gas on the piston is Ap, and the work done by this force while the piston is forced out through the distance PP′ is

$$Ap \, . \, \mathrm{PP'}.$$

But
$$A \, . \, \mathrm{PP'} = \text{volume of } \mathrm{PQQ'P'},$$
$$= v' - v.$$

Hence the work done $= p\,(v' - v)$,

which was to be proved.

In the same way we may show that the work done *on* a gas whose pressure is p, when the volume is reduced from v' to v at constant temperature, where $v' - v$ is infinitesimal, is also equal to $p\,(v' - v)$.

In applying this formula to a numerical example, care must be taken to express the pressure and the volume in appropriate units. If the work is to be expressed in foot-pounds, p must be expressed in pounds weight per square foot, and v and v' must be expressed in cubic feet.

Ex.—Taking the atmospheric pressure as a pressure of 15 lbwt. per square inch, find the number of foot-pounds of work done against atmospheric pressure when a piston accurately fitting the sides of a cylinder is drawn up to leave a vacuum of 12000 cubic inches.

Let A denote the section of the cylinder in square feet, and x the distance in feet through which the piston is drawn up, so that—

$$Ax = 12000/1728.$$

The pressure of the atmosphere is 15 lbwt. per square inch, or 15×144 lbwt. per square foot. Hence the constant pressure against which the

piston is raised is $15 \times 144 \times A$ lbwt. The work done against this pressure in raising the piston x feet is—

$$15 \times 144 \times A \times x \text{ foot-pounds.}$$

But since $Ax = 12000/1728$, the work done is

$$= 15 \times 144 \times 12000/1728,$$
$$= 15000 \text{ foot-pounds.}$$

127. Approximate Calculation of the Work done by a Gas during a Finite Expansion.

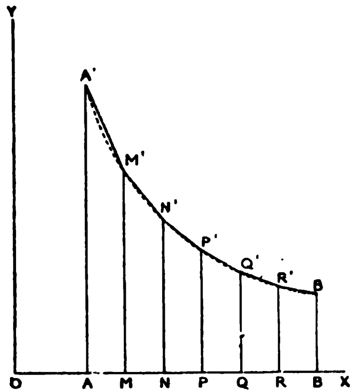
It follows from Boyle's Law that in the case in which the change of volume is finite, *i.e.* not very small, we cannot neglect the change of pressure. Hence the formula given in the preceding article for the work done during an infinitesimal expansion cannot be applied directly to calculate the work done when $v' - v$ is finite.

We may, however, obtain an *approximate* value for the amount of work done during a finite expansion from volume v to volume v' by supposing that the expansion takes place by a large number of stages, in each of which the increment of volume is very small. In each stage of the expansion the pressure may be taken to be constant, and equal to half the sum of the initial and final pressures during that stage. These pressures may be calculated from Boyle's Law, and by multiplying half their sum by the increment of volume we shall obtain a product which represents *approximately* the work done during that stage of the expansion. By adding up the amounts of work done during each of the stages of the expansion in succession, we shall obtain a sum which is *approximately* equal to the amount of work done during the expansion from the initial to the final volume.

This method of calculating the work done during a finite expansion is equivalent to the following construction for an area which represents approximately the amount of work done by the expanding gas.

Take OX and OY, two lines meeting at right angles at O; measure along OX lengths OA and OB to represent on any convenient scale the initial and final volumes of the gas; divide AB into a number of small elements AM, MN, NP...; and through

the points of division draw, perpendicular to AB, lengths AA′, MM′, NN′...BB′ to represent the pressure of the gas when the volume is OA, OM, ON...OB respectively. Draw the straight lines A′M′, M′N′... R′B′. Then the work done by the gas in expanding from the volume OA to the volume OB is *approximately* equal to the number of units of area in the figure bounded by AB, the ordinates AA′ and BB′, and the straight lines A′M′, M′N′,...R′B′.



To prove this it is sufficient to show that the area of the trapezium A′AMM′ represents the work done by the gas in expanding from the volume OA to the volume OM. It follows from the preceding article that the amount of work done in this expansion is equal to AA′ × AM on the supposition that the pressure is constant and equal to AA′, and is equal to MM′ × AM on the supposition that the pressure is constant and equal to MM′. We may take half the sum of the amounts obtained on these two suppositions as an *approximation* to the amount of work actually done by the varying pressure. Thus the amount of work done while the gas expands from the volume OA to the volume OM is approximately

$$= \tfrac{1}{2}\,AA' \times AM + \tfrac{1}{2}\,MM' \times AM,$$
$$= \tfrac{1}{2}\,(AA' + MM') \times AM,$$
$$= \text{area of the figure } A'AMM'.$$

A similar proof holds for the figures M′N, N′P..., and, therefore, the work done during the finite expansion from OA to OB is the sum of the areas of the figures A′M, M′N′... R′B, that is, is equal to the area bounded by AB, the ordinates AA′ and BB′, and the straight lines A′M′, M′N′,... R′B′.

Ex.—A cylinder, fitted with a piston, contains 8 cubic feet of compressed air at a pressure of 45 lbwt. per square inch. If the air expands until its volume is 10 cubic feet, find approximately the number of foot-pounds of work done during the expansion by the pressure of the air on the piston.

The following table shows corresponding values of volume and pressure. The volume, v, is expressed *in cubic feet*, and the pressure, p, is expressed in *lbwt. per square foot*:—

v	p
8	$45 \times 144, = 6480$
9	$\frac{8}{9} \times 6480, = 5760$
10	$\frac{8}{10} \times 6480, = 5184$

The work done in expanding from 8 cubic feet to 9 cubic feet is approximately

$$= \tfrac{1}{2}(6480 + 5760) \times 1,$$
$$= 6120 \text{ foot-pounds;}$$

and the work done in expanding from 9 cubic feet to 10 cubic feet is approximately

$$= \tfrac{1}{2}(5760 + 5184) \times 1,$$
$$= 5472 \text{ foot-pounds.}$$

Hence the work done in expanding from 8 cubic feet to 10 cubic feet is approximately

$$= 6120 + 5472,$$
$$= 11592 \text{ foot-pounds.}$$

128. Exact Formula for the Work done in a Finite Expansion.

It is evident that the smaller and smaller we take the elements of volume AM, MN,... the less and less will be the error committed in applying the approximate rule of the preceding article to find the work done in a finite expansion.

Now when the elements AM, MN,... become infinitesimal, the points A′, M′, N′,... will be consecutive points on a curve which is such that the abscissa and ordinate of any point on it represent corresponding values of the volume and the pressure. And, as in Art. 44, the area of this curve between the ordinates AA′ and BB′ will represent *exactly* the work done during the expansion from the volume OA to the volume OB.

Let v and p denote the abscissa OP and the ordinate PP′ of any point P′ on this curve. Then by Boyle's Law

$$vp = C,$$

where C is a number which is a constant during the expansion. This relation between v and p is the *equation to the curve* on which the points A′, M′, N′,... all lie when the increments of volume become infinitesimal.

This curve is represented in the figure by the dotted line. It is known from conic sections that the curve represented by the above equation is a rectangular hyperbola of which OX and OY are the asymptotes.

Let $OA = V$, and $OB = V'$, so that V and V' denote respectively the initial and final volumes of the gas; and let P denote the pressure when the volume is V. It is proved in the integral calculus that the area of the curve $vp = C$ between the ordi nates corresponding to $v = V$ and $v = V'$ is

$$C \times \text{hyp. log. } (V'/V),$$

where hyp. log. (V'/V) means the *hyperbolic* or *Napierian* logarithm of the fraction V'/V.

Assuming this result, it follows that the work done by a gas which occupies a volume V at pressure P, in expanding from volume V to volume V', is

$$PV \text{ hyp. log. } (V'/V) \ldots\ldots\ldots\ldots (1)$$

Since the logarithm of a fraction is equal to the logarithm of the numerator diminished by the logarithm of the denominator, this result may be written in the equivalent form

$$PV (\text{hyp. log. } V' - \text{hyp. log. } V) \ldots\ldots\ldots\ldots (2)$$

If we wish to use logarithms to base 10, we must divide the expressions (1) and (2) by μ, the modulus, a number which is approximately* equal to ·4. Thus we obtain an expression for the work done in the form

$$(PV/\mu) \log._{10} (V'/V) \ldots\ldots\ldots\ldots (3)$$

* The value of μ to 5 places of decimals is ·43429.

Ex. 1.—A quantity of air which occupies a volume of 8 cubic feet under a pressure of 45 lbwt. per square inch is allowed to expand to 10 cubic feet. Find the number of foot-pounds of work done by the expanding air.

Here
$$P = 45 \times 144, = 6480,$$
$$V = 8, \text{ and } V' = 10.$$

Hence the number of foot-pounds of work done

$$= 8 \times 6480 \text{ hyp. log. } 10/8.$$

From a table of hyperbolic logarithms it is found that the hyperbolic logarithm of 10/8, or 5/4, is ·2231436. Hence the work done in the expansion

$$= 8 \times 6480 \times \cdot 2231436,$$
$$= 11568 \text{ foot-pounds.}$$

By the approximate method it was found (see Ex., Art. 127) that the work done was 11592 foot-pounds. Hence the approximate method gives a result which is in this case less than 1/4 per cent in excess of the true value.

Ex. 2.—The quantity of air which occupies a volume of 50 cubic feet under a pressure of 15 lbs. per square inch is allowed to expand at a constant temperature to 60 cubic feet. Find the number of foot-pounds of work done.

[Hyp. log. 1·2 = 0·1823216.]

Here
$$P = 15 \times 144, = 2160,$$
$$V' = 60, \ V = 50,$$
$$\text{hyp. log. } (V'/V) = \text{hyp. log. } 1{\cdot}2, = \cdot 1823216.$$

Hence the amount of work done

$$= 2160 \times 50 \times \cdot 1823216$$
$$= 19691 \text{ foot-pounds.}$$

129. Elasticity of Fluids.

When a change of shape is produced in a fluid, the fluid exhibits no tendency to return to its original shape. This is expressed by saying that *fluids have no elasticity of shape.*

Fluids, however, in common with solids, possess the property of resisting change of volume. If a compression is produced in a fluid by the application of pressure, the fluid will return to its original volume when the pressure which produced the compression is removed, if the temperature is constant. This is expressed by saying that the *elasticity of volume of fluids is perfect.*

The pressure which produces a compression in a fluid is called the *stress*, and the number which measures the compression is called the *strain.*

The stress is measured in units of force per unit of area, *e.g.* in pounds weight per square inch. The strain is measured by the change of volume

per unit of volume. Thus if the volume of a fluid is diminished from v to v', the whole compression is $v - v'$, and the strain is $(v - v')/v$.

The value of the fraction

$$\text{stress}/\text{strain}$$

for a very small change of volume in any fluid is called the *coefficient of elasticity* of the fluid. The numerical value of this fraction, in the case of different fluids, will be greater the smaller the compression produced by a given stress, or, what is the same thing, this fraction will be greater the greater the resistance of the fluid to compression. Hence the coefficient of elasticity of a fluid measures the resistance which the fluid offers to compression.

In the case of liquids, the coefficient of elasticity must be found by experiment. Liquids offer great resistance to compression, and their coefficients of elasticity are therefore large. Thus, for example, the coefficient of elasticity of water is about 300,000 pounds weight per square inch.

In the case of gases, it can be shown from Boyle's Law that, on the supposition that the temperature is constant, the coefficient of elasticity is numerically equal to the pressure.

To prove this, let p denote the pressure and v the volume of a mass of gas. Let the pressure be increased to $p+k$, and in consequence let the volume be reduced to $v-h$; and let us suppose that k and h are very small quantities.

The strain $= h/v$,

and the stress producing this strain $= k$.

But by Boyle's Law

$$(p+k)\,(v-h) = pv,$$

from which

$$kv = hp + hk;$$

or

$$k \div \frac{h}{v} = p + k.$$

The last written equation shows that for the gas the fraction—stress/strain—is equal to $p+k$, the increased pressure of the gas, which, since k is very small, will be practically the same as the original pressure p.

Hence the resistance to compression of a gas is measured by the pressure of the gas.

Ex.—Assuming that mercury is compressed two-millionths of its volume by the application of a pressure of 14 lbwt. per square inch, find the coefficient of elasticity for mercury.

Here the strain $= 2/10^6$,

and the stress $= 14$ lbwt. per square inch.

Hence the coefficient of elasticity for mercury

$= 14 \div 2/10^6$,

$= 7$ million pounds weight per square inch.

EXAMPLES XVI.

[In all questions the temperature is supposed to be constant.]

(*The Answers are given on page* 337.)

A.

1. When a siphon manometer is in communication with a vessel containing compressed air, the surface of mercury in the open branch stands 12 inches higher than the surface in the other branch. Half the quantity of air in the vessel is removed by means of the air-pump, and it is found that the surface of mercury in the open branch falls through a height of $10\frac{1}{2}$ inches. Find the height of the barometer and the original pressure of air in the vessel.

2. In a compressed-air manometer the surfaces of the mercury in the two branches stand at the same level when the air confined in the closed branch is at a pressure of 30 inches of mercury, and the length of the column of air is 10 inches. How much will the surface of the mercury in the closed branch rise when the other branch is in communication with compressed air at a pressure of 120 inches of mercury?

3. A cylinder is fitted with a piston which works in it easily and air-tight. When the water barometer stands at 33 feet the piston is 1 foot above the bottom of the cylinder. If the cylinder were sunk slowly in water to a depth of 44 feet, how high would the piston (whose weight may be neglected) be above the bottom of the cylinder?

4. A cylinder, open at one end, whose diameter is 6 feet and height 10 feet, is immersed mouth downwards in water until the upper end is 20 feet below the surface. Find how high the water will rise inside the mouth of the cylinder, taking the height of the water barometer to be 33 feet.

If air is now pumped into the cylinder so as to completely expel the water, find what volume this air would occupy at atmospheric pressure.

5. Taking the standard atmospheric pressure as a pressure of a million dynes per square centimetre, find the work done in ergs against atmospheric pressure when a piston accurately fitting the sides of a cylinder is drawn up so as to leave a vacuum of 240 cubic centimetres.

6. A quantity of gas occupies a volume of 10 cubic feet when the pressure is 20 lbs. per square inch. If the gas expands until its volume is 50 cubic feet, find the number of foot-pounds of work done during the expansion.

[Hyp. log. $5 = 1{\cdot}60944$.]

7. A quantity of air occupies a volume of 20 cubic feet when the pressure is 40 lbs. per square inch. If the gas is allowed to expand until the pressure is reduced to 20 lbs. per square inch, find the number of foot-pounds of work done in the expansion.

[Hyp. log. $2 = {\cdot}69315$.]

8. A quantity of air occupies a volume of 24,000 cubic inches when its pressure is 14 lbs. per square inch. Find the number of foot-pounds of work done in compressing the air until its volume is reduced to 8000 cubic inches.

[Hyp. log. $3 = 1{\cdot}09861$.]

9. A quantity of oxygen in a vessel at atmospheric pressure is compressed until its pressure reaches 20 atmospheres, its volume being then 1/4 of a cubic foot. Find the number of foot-pounds of work done in the compression.

[Hyp. log. $20 = 2{\cdot}995732$.]

B.

10. A tube of uniform bore, stopped at one end, and of the same length, h, as the barometric column, is put vertically with its open end downward in mercury. At what depth must the open end be below the surface of the mercury if the mercury fills $(1/n)$th part of the tube?

11. Mercury is poured into a U-tube, open at both ends, of uniform sectional area 1 square centimetre, until n centimetres of each limb are left unfilled. One end is now stopped up, and n cubic centimetres of mercury are then poured into the other limb. If the difference of levels of the surfaces of the mercury in the two limbs is d centimetres, find the height of the barometer.

12. When a siphon manometer is in communication with a vessel containing compressed air, the surface of mercury in the open branch stands a inches higher than the surface in the other branch. Half the quantity of air in the vessel is removed by means of the air-pump, and it is found that the surface of mercury in the open branch falls through a height of b inches. Find the height of the barometer and the original pressure of air in the vessel.

13. Into a tube, which is closed at one end, a ounces of some solid are put, and then the tube is closed by a piston. It is observed that the pressure of air under the piston equals p when the total volume of the tube under the piston measures v cubic inches, while the pressure is p' for a volume v'. If 1 cubic inch of water weighs d ounces, find the specific gravity of the solid.

14. A cylinder containing compressed air is fitted with a piston which works in it easily and air-tight. The pressure of air is p, the area of the section of the cylinder is A, and the height of the piston above the bottom of the cylinder is a. If the piston is allowed to ascend through a distance c, find the work done in the expansion.

As a numerical example, take the case in which the area of the piston is 1000 square inches, the original volume is 12,000 cubic inches, the original pressure is 20 lbs. per square inch, and the final volume is 30,000 cubic inches.

[Hyp. log. $2 = {\cdot}69315$, hyp. log. $5 = 1{\cdot}60944$.]

15. Two barometers are placed side by side at the sea-level; one, which has a perfect vacuum at the top, stands at h inches; the other, which has an imperfect vacuum at the top, a inches long, stands at k inches. If it were possible for the force of gravity to change from g to g_1, other things remaining the same, what effect (if any) would this have on the height of the two barometers?

Obtain numerical results when h, k, and a are 30, 28, and 6 inches respectively, and g and g_1 are 32 and 24.

Chapter XIII.—LAW OF CHARLES. VAPOURS.

Measurement of Temperature, Arts. 130 to 134.

130. Definitions of Equal Temperatures, and of Higher and Lower Temperatures.

Def. Two bodies are said to be at *equal* temperatures, or at the same temperature, if on bringing them together there is no passage of heat from one to the other.

Def. If on bringing two bodies A and B together heat passes from A to B, then A is said to be at a *higher* temperature than B, and B at a *lower* temperature than A.

It would be inconvenient in most cases to compare the temperatures of two bodies by bringing them into contact, and thus directly applying the test for equal or unequal temperatures. In the practical determination of temperatures the bodies are brought into contact in succession with the same body—the *thermometer*. It is found by experiment that two bodies are at equal temperatures when they are each at the same temperature as a third body. On this experimental result is founded the method of measuring the temperatures of bodies by means of the thermometer.

The student must carefully distinguish between *heat* and *temperature*. Heat is the term applied to the form of energy whose passage into a body produces change of thermal condition, that is, change of temperature, and changes in the properties of the body. It was at one time believed that heat was a form of matter, but it is now established that heat is a form of energy which can be transformed into mechanical work. The study of the laws of transformation of heat into mechanical work forms the Science of Thermo-dynamics. In this book we shall be concerned only with the effect of a change of temperature on the volume and pressure of a gas.

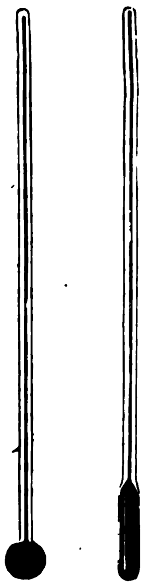
131. The Thermometer.

A thermometer is an instrument for determining the temperature of a body.

In general the volume of a body increases as its temperature rises. This

property of bodies is used in the measurement of temperature. Thermometers are constructed by means of which the temperature of any body is determined by the volume which a definite mass of mercury or alcohol occupies at the same temperature as the body.

The **mercurial thermometer** consists of a slender tube of glass, at one end of which is blown a bulb. The bulb and lower part of the stem contain mercury. In the construction of the instrument the mercury in the tube is boiled in order to expel the air, and the top of the tube is then closed.



Thermometers.

When a thermometer and a body are brought into contact, the temperature of the thermometer will rise or fall until it becomes equal to the temperature of the body. The higher is this temperature the greater will be the volume of the mercury in the thermometer, and therefore the higher will the mercury stand in the tube. Thus the point to which the mercury rises in the tube will be definite for a definite temperature, and will therefore serve to determine the temperature of the body with which the thermometer is in contact.

132. Determination of the Fixed Points of the Thermometer.

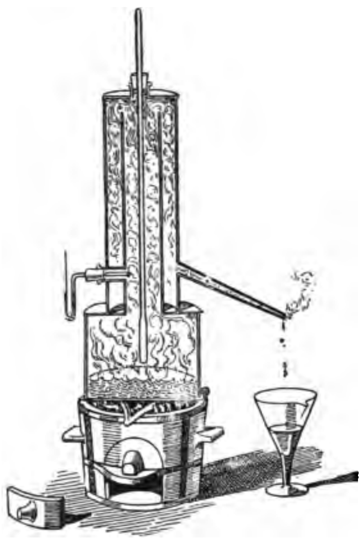
After the thermometer has been filled, the points on the stem at which the mercury stands, *first* when the temperature is that of melting ice, and *secondly* when the temperature is that of boiling water, are next determined by experiment.

Determination of the freezing point.—The thermometer is placed in a vessel with its bulb and part of its stem surrounded by melting ice and water, and a mark is made on the glass tube at the point of the stem to which the mercury rises. This marks the *freezing point* on the thermometer, that is, the point to which the mercury rises when the temperature is that at which water freezes.

Determination of the boiling point.—The thermometer is next immersed in the steam which issues from a vessel containing boiling water, as in the accompanying figure. A mark is made on the glass tube at the point where the surface of mercury in the tube now stands. This marks the

boiling point for the atmospheric pressure at the time of the experiment. We shall see that the temperature at which water boils varies with the atmospheric pressure, being higher the greater the pressure.

The boiling point and the freezing point are called the *fixed points of the thermometer*, but the boiling point is not a fixed point in the same sense as the freezing point, which is independent of the atmospheric pressure. When we speak hereafter of the boiling point without any mention of the atmospheric pressure, we shall always mean the temperature at which water boils when the atmospheric pressure is the pressure of the standard atmosphere of 760 millimetres, or 30 inches of mercury. If at the time of the determination of the boiling point of a thermometer the barometer does not stand at 30 inches, the mark on the stem will not correspond to the temperature of the boiling point at standard pressure, and an adjustment of the mark must be made.



133. Graduation of a Thermometer.

After the fixed points of the thermometer have been determined by experiment the thermometer is then graduated. The graduation may proceed according to one of three scales: (i) *Fahrenheit's scale,* (ii) *the Centigrade scale,* (iii) *Réaumur's scale.* Fahrenheit's scale is in common use in this country, and Réaumur's in some parts of the Continent. Physicists now express all experimental results in the Centigrade scale.

In Fahrenheit's scale the freezing point is marked 32 degrees (written 32°) and the boiling point 212°, the part of the tube between these two points being divided into 180 equal divisions or degrees.

In the Centigrade scale the freezing point is marked zero or 0°, and the boiling point 100°.

In Réaumur's scale the freezing point is marked zero or 0°, and the boiling point 80°.

In each of the three scales the graduation is continued below the freezing point and above the boiling point. The divisions below zero are distinguished by the sign minus (–). Thus –10° Centigrade means a temperature of 10 degrees below zero on the Centigrade scale.

134. Relations between the Three Scales.

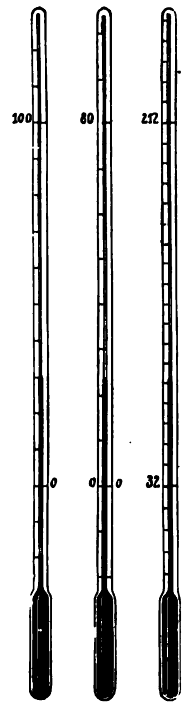
Since the interval between the fixed points in Fahrenheit's scale is 180°, in the Centigrade scale 100°, and in Réaumur's scale 80°, it follows that intervals of temperature in the three scales are connected by the following equations—

$$180^\circ \text{ Fahrenheit} = 100^\circ \text{ Centigrade} = 80^\circ \text{ Réaumur},$$

or,

$$9^\circ \quad ,, \quad = 5^\circ \quad ,, \quad = 4^\circ \quad ,,$$

Hence one degree Fahrenheit corresponds to 5/9 of a degree Centigrade and 4/9 of a degree Réaumur.



Cent. Réaum. Fahr.

To compare the *readings* of the *same* temperature in the three thermometers, let F, C, R denote the number of degrees in the three scales respectively; then $F-32$, C, R denote respectively for that temperature the number of divisions above the freezing point in the three scales. Now the length of the part of the tube between the freezing and boiling points must be divided in the same ratio in the three thermometers by the point where the mercury stands. Hence—

$$\frac{F-32}{180} = \frac{C}{10} = \frac{R}{80};$$

or

$$\frac{F-32}{9} = \frac{C}{5} = \frac{R}{4}.$$

From these equations we can express any two of the readings in terms of the third.

Thus to express F and R in terms of C we have—

$$F = 32 + \tfrac{9}{5}C, \quad R = \tfrac{4}{5}C. \; \ldots\ldots\ldots\ldots \quad (1)$$

We express C and R in terms of F by the equations—

$$C = \tfrac{5}{9}(F-32), \quad R = \tfrac{4}{9}(F-32) \ldots\ldots\ldots\ldots (2)$$

Lastly, F and C are found from R by the equations—

$$F = 32 + \tfrac{9}{4}R, \quad C = \tfrac{5}{4}R. \ldots\ldots\ldots\ldots \quad (3)$$

The student must bear in mind what is the exact significance of the number which measures the temperature of a body. The statement that the temperature of a body is 20° C. means nothing more than this, that the thermal condition of the body is the same as that of a Centigrade thermometer when the mercury in the thermometer stands at the reading 20 on the scale. Thus the number which measures the temperature of a body is a number to which corresponds a definite condition or state of the body with respect to heat.

Ex. 1.—Mercury freezes at $-39°$ C. Express the temperature in Fahrenheit's scale.

Here $C=-39°$, and therefore by the first equation of (1),

$$F=32+\tfrac{9}{5}(-39)=32-70\cdot2=-38°\cdot2.$$

Ex. 2.—What reading in the Centigrade scale corresponds to the zero in the Fahrenheit scale?

Here $F=0$, and we have to find C.

$$C=\tfrac{5}{9}(F-32)=\tfrac{5}{9}(-32)=-17°\cdot\dot{7}.$$

Ex. 3.—What temperature is represented by the same reading in the Fahrenheit and Centigrade scales? What is the corresponding reading in the Réaumur scale?

Here $F=C$, and substituting C for F in the first equation of (1) we get

$$C=32+\tfrac{9}{5}C,$$
$$\text{giving } C=F=-40°.$$

Substituting this value for C in the second equation of (1) we get

$$R=\tfrac{4}{5}(-40)=-32°.$$

EXAMPLES.

1. Reduce the following readings, which are on the Fahrenheit scale, to readings on the Centigrade and Réaumur scales respectively:—

(i) 176°; (ii) 158°; (iii) 131°; (iv) 95°; (v) 60°·8; (vi) 26°·6; (vii) $-1°\cdot4$; (viii) $-43°\cdot6$; (ix) $-49°$; (x) $-56°\cdot2$.

Ans. On the Centigrade scale:—(i) 80°; (ii) 70°; (iii) 55°; (iv) 35°; (v) 16°; (vi) $-3°$; (vii) $-18°\cdot56$; (viii) $-42°$; (ix) $-45°$; (x) $-49°$.

On the Réaumur scale:—(i) 64°; (ii) 56°; (iii) 44°; (iv) 28°; (v) 12°·8; (vi) $-2°\cdot4$; (vii) $-14°\cdot84$; (viii) $-33°\cdot6$; (ix) $-36°$; (x) $-39°\cdot2$.

2. Reduce the following Centigrade readings to Fahrenheit and Réaumur respectively:—

(i) 96°; (ii) 80°; (iii) 57°·3; (iv) 33°·4; (v) 18°·5; (vi) $-3°$; (vii) $-15°\cdot4$; (viii) $-27°\cdot5$; (ix) $-30°$; (x) $-49°$.

Ans. On the Fahrenheit scale:—(i) 204°·8; (ii) 176°; (iii) 135°·14; (iv) 92°·12; (v) 65°·3; (vi) 26°·6; (vii) 4°·28; (viii) $-17°\cdot5$; (ix) $-22°$; (x) $-56°\cdot2$.

On the Réaumur scale:—(i) 76°·8; (ii) 64°; (iii) 45°·84; (iv) 26°·72; (v) 14°·8; (vi) $-2°\cdot4$; (vii) $-12°\cdot32$; (viii) $-22°$; (ix) $-24°$; (x) $-39°\cdot2$.

3. Reduce the following Réaumur readings to Fahrenheit and Centigrade respectively:—

(i) 72°; (ii) 48°·8; (iii) $-37°\cdot6$.

Ans. On the Fahrenheit scale:—(i) 194°; (ii) 141°·8; (iii) $-52°\cdot6$.

On the Centigrade scale:—(i) 90°· (ii) 61°; (iii) $-47°$.

Relation between Volume, Pressure, and Temperature of a Gas, Arts. 135 to 138.

135. **Law of Charles.**

Suppose that a mass of air is enclosed in a cylinder, placed with its axis horizontal, and fitted with an air-tight piston which moves easily in the cylinder. If the temperature of the air is raised, the gas will tend to increase in volume, and will push out the piston against the constant pressure of the atmosphere. The air in the cylinder would be an example of a gas expanding, due to a rise of temperature *under constant pressure.*

It is found by experiment that when *any* gas is allowed to expand, *under constant pressure*, the increase in the volume due to a rise of temperature is subject to the following law, known as the **Law of Charles.***

The increase in the volume of any gas whose pressure is constant, for a rise of temperature of 1° C., is a constant fraction, = ·003665 *of the volume which the gas would occupy at 0° C.*

The ratio which the change of volume of a gas, kept under constant pressure, for a *change* of temperature of 1° C., bears to the volume which the gas would occupy at 0° C. is called the *coefficient of expansion* for gases. Thus the Law of Charles is equivalent to the following:—

The coefficient of expansion, under constant pressure, is the same for all gases and for all temperatures, and is numerically equal to ·003665.

The Law of Charles may be expressed in symbols. Let a denote the coefficient of expansion, so that $a = \cdot 003665, = 1/273$, V_t the volume of a quantity of a gas at the temperature $t°$ C., $V_{t'}$ the volume at the temperature $t'°$ C., and V_0 the volume at 0° C., the pressure of the gas being the same at all three temperatures; then shall

$$V_t = V_0 (1 + at) \qquad (1)$$

and

$$V_t / V_{t'} = (1 + at) / (1 + at') \qquad (2)$$

* This law of gases is sometimes assigned to *Gay-Lussac* and sometimes to *Dalton*. We shall always refer to it as the Law of Charles.

For, due to a rise of temperature of 1° C., the volume at the freezing temperature, V_0, is increased by the amount aV_0, and therefore, for a rise of temperature of t degrees the volume at freezing temperature is increased by atV_0. Hence

V_t = volume at freezing temperature
+ increase of volume for a rise of temperature of t degrees,
$= V_0 + atV_0, = V_0 (1 + at)$,

which is formula (1).
Similarly

$$V_{t'} = V_0 (1 + at');$$

and therefore by division

$$V_t / V_{t'} = (1 + at) : (1 + at'),$$

which is formula (2).

Ex. 1.—The volume of a mass of air at freezing temperature is 1000 cubic inches. What will be the volume of this air when the temperature rises to 15° C., the air being allowed to expand at constant pressure?

Here $V_0 = 1000$, $t = 15$, and it is required to find V_{15}.

From formula (1)

$$V_{15} = 1000 \; (1 + 15 \times \cdot 003665),$$
$$= 1055 \text{ cubic inches very nearly.}$$

Ex. 2.—If a given quantity of gas expands under a constant pressure in consequence of its temperature being raised from 20° C. to 21° C., find what ratio the increase of volume bears to its volume at 20° C.

If V_0 denote the volume of the gas at 0° C., then the volume of the gas at 20° C. is

$$V_0 \; (1 + 20 \times \cdot 003665);$$

and the *increase* in the volume when the temperature rises from 20° to 21° C. is

$$V_0 \times \cdot 003665.$$

Hence the required ratio

$$= \frac{V_0 \times \cdot 003665}{V_0 \; (1 + 20 \times \cdot 003665)}, \quad = \frac{\cdot 003665}{1 \cdot 0733},$$

$$= \frac{733}{214660}.$$

Ex. 3.—The volume of a quantity of air is 1200 cubic inches at the temperature 10° C. What will be its volume at − 10° C., the pressures being the same?

Here we make use of formula (2), putting

$$t=-10,\ t'=10,\ V_{t'}=1200.$$

We thus obtain the equation

$$\frac{\text{volume at }-10^\circ\text{ C.}}{1200}=\frac{1-10\times{\cdot}003665}{1+10\times{\cdot}003665},$$

$$={\cdot}96335/1{\cdot}03665;$$

from which we get

$$\text{volume at }-10^\circ\text{ C.}=1200\times{\cdot}96335/1{\cdot}03665,$$
$$=1115\text{ cubic inches nearly.}$$

136. Change of Volume of a Gas due to Change of Temperature and Pressure.

Given that the volume of a gas is V when its temperature is t degrees C. and its pressure is P, it is required to find its volume, V' say, when the temperature is changed to t' degrees C. and the pressure to P'.

This problem is solved by applying the Law of Boyle and the Law of Charles. From these laws we shall deduce the formula—

$$\frac{V'P'}{1+\alpha t'}=\frac{VP}{1+\alpha t},$$

in which α represents the coefficient of expansion for gases. By this formula V' may be found when V, P, t, P', and t' are all given.

To prove this formula let us consider in the first place what the volume V will become when the temperature is changed from t to t', *the pressure being maintained equal to* P. By formula (2) of the preceding article the volume will be changed from V to V_1, where

$$V_1=V(1+\alpha t')\ /\ (1+\alpha t).$$

Next, *keeping the temperature equal to* t', let the pressure be changed from P to P'. By Boyle's Law the volume will be changed in the inverse ratio of the pressures; that is, the volume will become V_1P/P'. But the gas will now be at the pressure P' and temperature t', and its volume is by supposition then equal to V'.

Hence

$$V' = V_1 P/P',$$
$$= VP\,(1 + at') \;/\; P'\,(1 + at);$$

from which

$$\frac{V'P'}{1 + at'} = \frac{VP}{1 + at},$$

the required formula.

137. Absolute Temperatures on the Air Thermometer.

We have seen that when the pressure of a mass of air is kept constant, the air being allowed to expand or contract as its temperature rises or falls, the volumes of the air will be definite for definite temperatures. Thus the changes in the volume of the air might be used to determine changes of temperature. This is the principle on which constant-pressure air thermometers are constructed.

If we suppose that the air in an air thermometer remains in the form of a gas at all temperatures, and that the volumes at all temperatures obey the Law of Charles, it would follow that as the temperature is reduced lower and lower the volume would be diminished without limit. Hence, *on the suppositions referred to*, at a certain temperature the volume would be zero, a result which is, of course, impossible. This temperature is called *Absolute Zero on the air thermometer.*

To determine what would be the Centigrade reading corresponding to absolute zero, we use the equation (1) of Art. 135—

$$V_t = V_0\,(1 + at),$$

in order to find what is the value of t when V_t is equal to zero. We thus get

$$V_0\,(1 + at) = 0,$$

from which, since V_0 is not zero,

$$1 + at = 0,$$

or $$t = -1/a.$$

Since $a = 1/273$, this gives $t = -273°$ C. Hence the absolute

zero of the air thermometer would be a temperature of $-273°$ on the Centigrade thermometer.

Temperatures measured from absolute zero, defined above, are called *Absolute Temperatures on the air thermometer.* If t denotes a temperature measured on the Centigrade scale from freezing point, and T denotes the same temperature measured on the same scale from absolute zero, then

$$\begin{aligned} T &= t + 1/a, \\ &= t + 273. \end{aligned}$$

The temperature of $-273°$ C. is much lower than the lowest temperature yet attained. Researches made in recent years have shown that all gases, including air and the other gases formerly known as *permanent* gases, change their state and become liquid, under the application of great pressure, at temperatures far above that of absolute zero.

138. The product of the Volume and Pressure of a given Mass of a Gas is proportional to the Absolute Temperature.

If t and t' denote two temperatures measured from the freezing point, and T and T' denote the corresponding absolute temperatures, all measured on the Centigrade scale, then

$$T = 1/a + t,$$

and

$$T' = 1/a + t'.$$

Therefore

$$\begin{aligned} T : T' &= 1/a + t : 1/a + t', \\ &= 1 + at : 1 + at'. \end{aligned}$$

Hence, by substituting T'/T for $(1 + at')/(1 + at)$ in the formula of Art. 136, we obtain

$$\frac{V'P'}{T'} = \frac{VP}{T}.$$

This formula shows that if V is the volume, P the pressure, and T the absolute temperature of a mass of a gas, then VP/T is constant.

We may write this result in the forms—

$$VP/T = C, \quad VP = CT,$$

where C represents a constant quantity.

Hence the product of the volume and pressure of a given mass of a gas is proportional to the absolute temperature.

The equation $VP/T = C$ is deduced from the Law of Boyle and the Law of Charles, and is the expression of these two laws in symbolical language. For, if we suppose T to be constant, the equation becomes $VP =$ a constant; and this is the Law of Boyle. If we suppose P to be constant, the equation becomes $V/T =$ a constant; and this is the Law of Charles.

The equation shows that if V is constant, then P/T is constant, or *the pressure of a gas, whose volume is constant, varies as the absolute temperature.*

Ex. 1.—A mass of air occupies a volume of 800 cubic inches at a temperature of 15° C. and a pressure of 30 inches of mercury. What will be the volume of this air at a temperature of 5° C. and at a pressure of 29·5 inches of mercury?

In the original condition of the air, the volume, V, was 800, the pressure, P, was 30, the absolute temperature, T, was $273 + 15$, $= 288$, and the problem is to find the volume, x say, when the pressure is 29·5, and the absolute temperature $273 + 5$, $= 278$. It is convenient to make a table of corresponding values of volume, pressure, and temperature, as follows:—

V	P	T
800	30	288
x	29·5	278

In the original condition of the air the value of VP/T was $800 \times 30/288$, and in the altered condition the value is $x \times 29{\cdot}5/278$; and these are equal.

Hence

$$\frac{x \times 29{\cdot}5}{278} = \frac{800 \times 30}{288},$$

from which $x = 785{\cdot}3$ cubic inches,

the volume required.

Ex. 2.—A cubical vessel, whose edge is 1 foot, is made air-tight when the barometer stands at 30 in. and the temperature of the air is 15° C.; if the temperature of the air is raised to 60° C. what is the increase of the pressure of the contained air on each face of the cube?

[A cubic inch of mercury may be taken to weigh half a pound.]

Here the volume is constant, and therefore the pressure varies as the ab-

solute temperature. Let P denote the pressure of the air in the vessel, in inches of mercury, at the temperature of 60° C. Then

$$\frac{P}{30}=\frac{273+60}{273+15}, \quad =\frac{37}{32};$$

from which
$$P=30\times 37/32,$$
$$=34{\cdot}6875.$$

Hence the *increase* of pressure is equal to a pressure of $(34{\cdot}6875-30)$, $=4{\cdot}6875$ inches of mercury.

Since a cubic inch of mercury weighs half a pound, a height of 4·6875 inches of mercury corresponds to a pressure of $\frac{1}{2}\times 4{\cdot}6875$, $=2{\cdot}34375$ lbwt. per square inch.

Therefore the increase of pressure on each face of the cube is

$$=2{\cdot}34375\times 144,$$
$$=337{\cdot}5 \text{ pounds weight.}$$

EXAMPLES.

[Take $\alpha=1/273$.]

1 A certain quantity of air occupies a volume of 1000 cubic inches at temperature 0° C. Find its volumes at the following temperatures, the pressure being constant:—

(i) 10° C. (ii) 18° C. (iii) 100° C. (iv) $-40°$ C.

Ans. In cubic inches—(i) 1036·6. (ii) 1066. (iii) 1366·3. (iv) 853·5.

2. The volume of a certain quantity of a gas is 1500 cubic inches at a pressure of 30 inches of mercury. If the temperature is constant, find what volumes the gas will occupy respectively under the following pressures, expressed in inches of mercury:—

(i) 28·75. (ii) 29·5. (iii) 30·75.

Ans. In cubic inches—(i) 1565·2. (ii) 1525·4. (iii) 1463·4.

3. The volume of a certain quantity of a gas at temperature 10° C. and pressure 30 inches of mercury is 350 cubic inches. Find the volumes at the following temperatures and pressures:—

(i) Temperature, 15° C.; pressure 28·75 in. of mercury.
(ii) „ 50° C.; „ 30·75 „
(iii) „ $-25°$ C.; „ 30·5 „

Ans. In cubic inches—(i) 371·7. (ii) 389·7. (iii) 301·7.

4. The volumes of a certain mass of air at temperature 10° C. and 20° C. are equal. Compare the pressures.

Ans. As 283 is to 293.

5. The volumes of a quantity of gas at 75° C. and $-5°$ C. are equal. Compare the pressures.

Ans. As 87 is to 67.

6. The volumes of a certain quantity of air at pressures of 29·5 and 30 inches of mercury are equal. If the temperature at the lower pressure is 18° C., find the temperature at the higher pressure.

Ans. 23° C. very nearly.

Vapours, Arts. 139 to 147.

139. Formation of Vapour.

Vapours are the gases into which liquids are converted either in the process of *evaporation* or in the process of *ebullition* or boiling.

Evaporation proceeds silently, and in general slowly. Liquids evaporate at all ordinary temperatures, the rate of evaporation being more rapid the higher the temperature. For the same temperature there are great variations in the rates of evaporation of different liquids. Some liquids, such as ether and alcohol, are very *volatile,* that is, pass readily into the state of vapour; while in some other liquids, such as mercury and sulphuric acid, the rate of evaporation is very slow.

In *ebullition*, bubbles of vapour of the liquid are formed in the liquid, and rise to the surface. The temperature at which ebullition occurs is always the same for the same liquid, as long as the atmospheric pressure is constant. Thus when the atmospheric pressure is that of 760 millimetres, or 30 inches of mercury, water boils at the temperature of 100° C., and mercury at the temperature of 353° C. We shall see that the higher the atmospheric pressure, the higher is the temperature at which a given liquid boils.

When heat is applied to a mass of liquid, the temperature of the liquid rises, and the rate of evaporation is increased. The rise of temperature continues until the boiling point of the liquid is reached, after which the liquid will pass rapidly into the state of vapour, and the temperature will remain constant. Thus the boiling point of a liquid at a given atmospheric pressure is the temperature above which it is impossible, at that pressure and under ordinary circumstances, to raise the temperature of the liquid.

The formation of vapour is illustrated in the well-known phenomena of the evaporation of water and of the conversion of water into steam. Water evaporates at all temperatures, so that the atmosphere always contains a greater or less amount of vapour of water. Steam is aqueous vapour whose temperature is equal to or higher than the temperature of boiling water.

Aqueous vapour is invisible. The white cloud which issues from a vessel

containing boiling water is composed of particles of liquid into which the steam rising from the vessel is condensed on passing into a space whose temperature is lower than that of boiling water.

140. Vapour in contact with the Liquid producing it.

If we introduce into a closed vessel a quantity of liquid which does not completely fill the vessel, evaporation of some of the liquid will take place. It is found that a certain definite quantity of vapour is formed, the amount depending on the volume and temperature of the space not occupied by the liquid.

When the maximum amount of vapour has been formed, the space above the liquid is said to be *saturated with vapour of the liquid*, and the vapour is said to be in the saturated condition.

For a given temperature the quantity of vapour of a given liquid required to saturate a space varies as the volume of the space. If the volume of the space is kept constant, the quantity of vapour required to saturate the space is greater the higher the temperature. Hence for a given temperature a maximum density of the vapour of a given liquid may be reached, this density being greater the higher the temperature. Saturated vapour at any temperature is therefore said to be *vapour at its maximum density for that temperature.*

If a space contains saturated vapour, and if, without change of temperature, the volume of the space is reduced, some of the vapour will be condensed into liquid, and the density of the vapour which remains uncondensed will not be altered. Also, if the temperature of the space is lowered while the volume is kept constant, some of the vapour will be condensed, the space remaining saturated with vapour at the lower temperature.

From these experimental results we conclude as follows:—

(i) *A vapour, contained in a closed space and in contact with the liquid producing it, remains at the maximum density for the temperature of the space.*

(ii) *If, without change of temperature, the volume of the space is increased, more vapour will be formed, and if the volume of the*

space is diminished, some of the vapour will be condensed, the density of the vapour remaining unchanged in both cases.

(iii) *If the volume of the space is kept constant, and the temperature raised, more vapour will be formed, so that the density of the vapour will be increased. If the temperature is lowered, some of the vapour will be condensed into liquid, so that the density of the vapour will be diminished.*

Superheated vapour is vapour of a liquid whose density is less than the density of saturated vapour of the same liquid at the same temperature.

Saturated vapour, which is enclosed in a vessel containing none of the liquid from which the vapour is produced, becomes superheated when the temperature is raised or the volume increased. On the other hand, superheated vapour may become saturated in consequence either of a fall of temperature or of a diminution of the volume of the space containing the vapour.

It is now known that air and the other gases which were formerly known as "permanent" gases, are vapours of liquids. All the permanent gases have been liquefied by the application of very great pressures at very low temperatures.

141. Pressure of Vapours.

A vapour differs from a permanent gas in this respect only —that it can more easily be condensed into the condition of a liquid. Thus a vapour exerts pressure on the sides of the vessel which contains it, and, as long as it remains in the condition of a gas, obeys the two laws of gases—the Law of Boyle and the Law of Charles.

When the temperature is constant, the pressure varies as the density, and is therefore greatest when the vapour is in the saturated condition. Hence—

(i) *For the vapour of a given liquid at a given temperature a certain definite pressure may be found, which is the maximum pressure of the vapour.*

The effect of reducing the volume of a space containing saturated vapour, without changing the temperature, is not to increase the pressure, but to condense part of the vapour, the pressure of the part that remains after condensation being the same as the original pressure.

(ii) *The maximum pressure of the vapour of a given liquid is greater the higher the temperature.*

The maximum pressure of a vapour at a given temperature must be determined by experiment. The following table gives the maximum pressures of the vapours of water and ether for the temperatures of 0°, 10°, 100°, and 120° on the Centigrade scale respectively, the pressures being expressed in inches of mercury:—

Temperature on the Centigrade Scale.	Maximum pressure of vapour of water in inches of mercury.	Maximum pressure of vapour of ether in inches of mercury.
0°	·18	7·3
10°	·36	11·3
100°	30	195
120°	60·5	304

When a vapour at any temperature is in the condition of non-saturated or superheated vapour, the pressure of the vapour bears to the maximum pressure for the same temperature the ratio of the density of the vapour to the density of saturated vapour at the same temperature. [Boyle's Law.] The density of superheated vapour is usually expressed as a fraction of the density of saturated vapour at the same temperature. For example, if the vapour is half saturated, its pressure is half the maximum pressure.

In general, if r is the ratio of the density of a superheated vapour to the maximum density at the same temperature, and if f is the maximum pressure for that temperature, then the pressure of the superheated vapour is rf.

The so-called "vacuum-space" above the mercury in the Torricellian experiment (Art. 108) is not a vacuum, but is filled with the vapour of mercury, which exerts a downward pressure on the column of mercury in the tube. Mercury is not a volatile liquid, and the maximum pressure of its vapour at ordinary temperatures is very small. The depression of the column due to the pressure of vapour of mercury in a barometer tube may therefore be neglected.

If we introduce a very small quantity of some volatile liquid, such as ether or alcohol, into the tube of the barometer, the liquid will rise to the surface of the mercury, and will there be immediately converted into vapour. The pressure of this vapour will produce a measurable depression

of the mercurial column. When more liquid is added, the depression increases until a certain point is reached, after which the addition of liquid ceases to have any additional effect in depressing the column, and the added liquid remains unevaporated. When this point is arrived at, the space above the mercury is filled with saturated vapour, which is exerting the maximum pressure of the vapour of the liquid for the existing temperature. From the above table we see that this pressure for vapour of ether at 10° C. is a pressure of 11·3 inches of mercury, and consequently the column of mercury would be depressed by this amount at 10° C. by the introduction of a quantity of ether sufficient to saturate the vacuum space above the mercury. If the temperature of the vacuum space is raised, and more liquid added if necessary, more vapour will be formed, and the pressure will increase, the increase of pressure producing an additional depression of the mercurial column.

The accompanying figure illustrates the effect of introducing a small quantity of a volatile liquid into a Torricellian vacuum. A long tube, *ab*, open at one end and closed at the other, is filled with mercury, and placed in a vertical position with its open end under the surface of mercury in a vessel. A small quantity of a volatile liquid is introduced into the vacuum space above the mercury, and a depression of the column of mercury immediately follows. The depression may be shown by inverting in the vessel another tube—in the figure the tube on the right—containing mercury, the space above which is a vacuum. The difference between the heights of the columns of mercury in the two tubes is equal to the pressure of the vapour of the liquid introduced into the tube *ab*.

142. Boiling Point of a Liquid.

A liquid whose surface is freely exposed to the atmosphere, boils at the temperature at which the maximum pressure of its vapour is equal to the

atmospheric pressure. As the maximum pressure of a vapour increases with the temperature, it follows that the greater the atmospheric pressure the higher is the temperature at which a given liquid boils.

We have seen that the greater the height above the sea-level, the less is the atmospheric pressure. It follows that the higher a station is above sea-level, the lower is the temperature at which water boils. For example, while at the sea-level water boils at 100° C., at Quito, which is 9200 feet above sea-level, the temperature at which water boils is 90° C.

143. Pressure of a Mixture of Gas and Vapour.

The following laws regarding a mixture of gas and vapour were discovered by Dalton. The second law is an extension to a mixture of gas and vapour of the law of a mixture of gases (Art. 121).

(i) *The maximum quantity of vapour which a given space can contain at a given temperature is the same whether the space is a vacuum or is occupied by a gas.*

(ii) *The pressure in a space containing a mixture of gas and vapour is the sum of the pressures which would be exerted by each of the constituents of the mixture, if the other constituent were not present in the space.*

Thus if a space contains a gas at pressure p, and if the space then becomes saturated with the vapour of any liquid, the pressure of the mixture will become equal to

$$p+f,$$

where f is the maximum pressure of the vapour at the temperature of the space.

If the vapour is not saturated, the pressure of the space will be

$$p+rf,$$

where r denotes the ratio of the density of the vapour to the maximum density at the temperature of the space.

It is found by experiment that due to a change in the volume and temperature of a space containing a mixture of gas and vapour, there is the same change in the pressure due to the gas as there would be if the vapour were not present. Hence the pressure due to the gas under the altered conditions may be found from the formula $PV/T=C$, which embodies the Law of

Boyle and the Law of Charles. Also, *if no condensation of the vapour takes place due to the change*, the pressure due to the vapour will be altered according to the same laws, and may be found by applying the same formula. The pressure of the mixture under the altered conditions will be the sum of the altered pressures due to the gas and the vapour respectively.

Ex. 1.—A space contains a gas at a pressure of 30 inches of mercury. The space then becomes saturated with vapour of a liquid, and it is then found that the pressure is that of 32 inches of mercury. If, without change of temperature, the volume of the space is reduced by a half, what will then be the pressure of the mixture?

When the space is reduced the pressure due to the presence of the gas will be doubled, and will therefore be equal to the pressure of 60 inches of mercury. Due to the change of volume some of the vapour will be condensed, and the pressure due to the remainder will be equal to the pressure of the vapour before condensation, that is, will be a pressure of 2 inches of mercury. Hence the pressure of the mixture after the change of volume will be a pressure of $60+2, =62$ inches of mercury.

The student will notice that we have left out of account the volume occupied by the liquid formed by the condensation of part of the vapour. In general this volume would be very small compared with the volume of the space.

Ex. 2.—A six-gallon boiler contains air as well as water. Heat is applied, and the pressure-gauge observed when there are exactly five gallons of water and one gallon of air and steam in the vessel. A pint of water is then drawn off, the temperature remaining unchanged, and it is observed that the pressure immediately falls to 17/18 of its former value. Prove that the pressures of the air and steam before the water was drawn off were equal.

Before and after the water was drawn off, the space above the water was filled with air and saturated steam (Art. 140). Let p and f denote respectively the pressure due to the air and the pressure due to the steam before the water was drawn off. The pressure of the mixture was then equal to $p+f$. The volume of the mixture was at first 1 gallon or 8 pints, and after the water was drawn off the volume was 9 pints. Hence the pressure due to the air after the water was drawn off was $8p/9$, and the pressure due to the steam was still equal to f, so that the pressure of the mixture was then equal to $8p/9+f$. But from the data of the problem this was equal to 17/18 of the original pressure of the mixture. Hence

$$8\,p/9+f=17\,(p+f)\,/18,$$

or

$$16\,p+18\,f=17\,p+17\,f,$$

from which

$$f=p.$$

Hence the pressures of the air and steam before the water was drawn off were equal.

144. The Dew-point.

At all temperatures the atmosphere contains aqueous vapour or moisture, the amount of which varies from place to place and from time to time. The atmosphere is said to be very moist when the amount of aqueous vapour in a given volume of air approaches to the amount required to saturate that volume at the existing temperature of the air. When the amount is only a small fraction of the amount required for saturation, the air is said to be very dry.

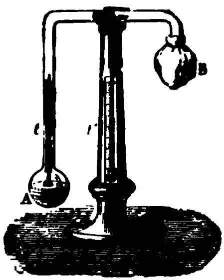
When the air is nearly saturated with moisture, a very slight fall of temperature will produce a condensation of some of the vapour. When the air is very dry, condensation of moisture will not begin until the temperature falls considerably below the existing temperature of the air.

The *dew-point* in a given state of the atmosphere is the temperature at which condensation of the moisture in the air would begin. When the air is nearly saturated with moisture, the dew-point is a temperature which is very little lower than the actual temperature of the air; but when the air is very dry, the dew-point is many degrees lower than the actual temperature.

Air which is in contact with a surface whose temperature is lower than the dew-point will be cooled below the dew-point, and in consequence some of the moisture in this air will be condensed and deposited on the surface. It is on this principle that the deposition of dew may be explained. After sunset there is often a considerable fall, due to radiation of heat, in the temperature of the surface of the earth, and the stratum of air in contact with the surface becomes cooled to a temperature which is lower than the dew-point. When this happens, some of the moisture in this stratum of air is condensed, and is deposited on the surface of the earth in the form of *dew.*

Hygrometers are instruments for determining the dew-point. In Daniell's hygrometer two glass globes A and B (see figure), which contain no air, are in communication, the globe A containing a quantity of ether, so that the space in the instrument not occupied by liquid ether is filled with vapour of ether. Covering the globe B is a piece of muslin kept wet with ether, the

rapid evaporation of which is accompanied by a fall of temperature and a condensation of some of the vapour of ether inside the globe B. This is followed by the rise of more vapour from the surface of the liquid in the globe A, and this by a fall in the temperature of that globe, in consequence of which the temperature of the air surrounding the globe falls lower and lower, and finally dew is deposited on the outside of the globe. The temperature of the globe A is indicated by a thermometer inside the globe, and the reading when dew begins to be formed is the dew-point.



145. Density of Dry Air.

The following is an *experimental result*:—

A cubic foot of dry air, that is, of air that contains no moisture, at the standard pressure of 30 inches of mercury and at the freezing temperature, weighs 566 grains.

Assuming this result, we may calculate the weight of a cubic foot of dry air at the pressure of P inches of mercury and temperature t° C. We have only to find the volume which this quantity of air would occupy at the pressure of 30 inches of mercury and at the freezing temperature. Let this volume be denoted by V'. Then the weight of a cubic foot of dry air at the pressure P and temperature t is 566 V' grains.

To find V' we use the formula (Art. 136),

$$\frac{V'P'}{1+at'}=\frac{VP}{1+at},$$

putting $P'=30,\ t=0,\ V=1.$

Thus we obtain

$$V'=\frac{P}{30(1+at)}.$$

Hence the weight of a cubic foot of dry air at the pressure P and temperature t

$$=566V',$$

$$=\frac{566P}{30(1+at)}\text{ grains.}$$

In expressing the density of air it is usual to use the metric

system. Corresponding to the above experimental result we have the following:—

A litre, $= 1000$ cubic centimetres, of dry air at the pressure of 760 millimetres of mercury and at the freezing temperature, weighs 1·293 grammes. Hence it follows that the weight of a litre of dry air at the pressure of P millimetres of mercury and at the temperature $t°$ C. is

$$\frac{1{\cdot}293\, P}{760(1 + at)} \text{ grammes.}$$

146. Density of Aqueous Vapour.

The following is an *experimental result:*—

The density of aqueous vapour is 5/8 of the density of dry air at the same temperature and the same pressure.

Thus we may find the weight of a given volume of moisture at a given temperature and a given pressure, by multiplying the mass of the same volume of dry air at the same temperature and the same pressure by the fraction 5/8.

Hence if p denote the pressure expressed in inches of mercury, of a quantity of aqueous vapour, and t the temperature on the Centigrade scale, the weight of a cubic foot of the vapour is

$$\frac{5}{8} \times \frac{566\, p}{30(1 + at)} \text{ grains.}$$

If the metric system is used, so that p denotes the pressure in millimetres of mercury, then the weight of a litre of the vapour is

$$\frac{5}{8} \times \frac{1{\cdot}293\, p}{760(1 + at)} \text{ grammes.}$$

It is usual to specify the pressure of aqueous vapour as a fraction of the maximum pressure at the temperature equal to that of the vapour. If f denote the maximum pressure for the temperature of the vapour, and r the fraction which the actual pressure bears to this maximum pressure, then the actual pressure is rf, and the formulæ given above may be modified by writing rf for p.

147. Density of Moist Air.

From the results of Arts. 145 and 146 we can construct a formula for calculating the weight of a given volume of *moist* air, that is, of a mixture of air and aqueous vapour.

Let P denote the pressure in inches of mercury of the mixture and t the temperature on the Centigrade scale, and let p denote the pressure in inches of mercury of aqueous vapour in the mixture. Then it follows from Dalton's Law (Art. 143) that the pressure due to the dry air in the mixture is $P-p$.

In order to calculate the weight of a cubic foot of the mixture, let us imagine that the moisture in this cubic foot is separated from the dry air, and introduced into a vacuum space whose volume is one cubic foot. Thus, instead of a cubic foot of moist air at pressure P and temperature t, we should have a cubic foot of dry air at pressure $P-p$ and temperature t, *and* a cubic foot of aqueous vapour at pressure p and temperature t. The sum of the weights of the dry air and the aqueous vapour will be equal to the weight of a cubic foot of the moist air.

The weight of the dry air is

$$\frac{566\,(P-p)}{30\,(1+at)} \text{ grains,} \qquad \text{(Art. 145)}$$

and the weight of the aqueous vapour is

$$\tfrac{5}{8} \times \frac{566\,p}{30\,(1+at)} \text{ grains.} \qquad \text{(Art. 146)}$$

Hence the weight of a *cubic foot* of the moist air is

$$\frac{566\,(P-p)}{30\,(1+at)} + \tfrac{5}{8} \times \frac{566\,p}{30\,(1+at)},$$

$$= \frac{566\,(P-\tfrac{3}{8}p)}{30\,(1+at)} \text{ grains} \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

It may be shown in the same way that, if P and p are expressed in millimetres of mercury, the weight of a *litre* of moist air

$$= \frac{1{\cdot}293\,(P-\tfrac{3}{8}p)}{760\,(1+at)} \text{ grammes.} \quad \ldots\ldots\ldots\ldots(2)$$

In using these formulæ to find the density of air in any state of the atmosphere, P must be taken to be the actual pressure of the atmosphere as measured by the height of the barometer. The pressure p will denote the pressure of aqueous vapour in the atmosphere. It can be shown* that in any state of the atmosphere the pressure of the aqueous vapour actually present in the air is equal to the maximum pressure of aqueous vapour at the dew-point. Tables of maximum pressure of aqueous vapour at different temperatures have been constructed on the results of experiments. It follows that when the dew-point has been found by observation (Art. 144), the value of p, the pressure of aqueous vapour present in the atmosphere, may be found by referring to a table of vapour pressures for the maximum pressure of aqueous vapour at the dew-point.

Ex. 1.—Find the weight in grammes of 2000 cubic centimetres of dry air at the temperature of 16° C. and pressure of 749 millimetres of mercury.

A volume of 2000 cubic centimetres is equal to 2 litres. Hence by Art. 145 the weight of the air

$$= \frac{2 \times 1{\cdot}293 \times 749}{760\,(1+16 \times {\cdot}003665)},$$
$$= 2{\cdot}407 \text{ grammes}.$$

Ex. 2.—Find the weight in grains of a cubic yard of air saturated with moisture at the temperature of 25° C. and pressure of 29·5 inches of mercury; being given that the maximum pressure of aqueous vapour at the temperature of 25° C. is ·94 of an inch of mercury.

Here we use the formula (1) given above for the weight of a cubic foot of moist air, putting in that formula

$$P = 29{\cdot}5,\ p = {\cdot}94,\ t = 25.$$

Hence the weight of a cubic yard, $=9$ cubic feet, of the air

$$= \frac{9 \times 566\,(29{\cdot}5 - \frac{3}{8} \times {\cdot}94)}{30\,(1+25 \times {\cdot}003665)},$$
$$= 4533{\cdot}8 \text{ grains}.$$

* Let p and f denote the pressures of the dry air and the moisture respectively in any state of the atmosphere, and let p' and f' denote the corresponding pressures when the temperature is reduced to the dew-poiht. We may suppose p' and f' to be the pressures for the moist air in contact with the globe A of Daniell's hygrometer (Art. 144) at the moment that the deposition of dew on this globe begins. Since this air is in equilibrium with the surrounding air, it follows that $p'+f'=p+f$.

It follows from the Laws of Gases that, due to this fall of temperature, p' is equal to kp and f' is equal to kf, where k is a number which depends only on the final and initial densities of the air, and on the dew-point and the actual temperature of the air. Hence $p'+f'=kp+kf$, $=k(p+f)$. But $p'+f'=p+f$, and therefore $k\,(p+f)=p+f$, showing that $k=1$. Hence $f'=f$, which was to be proved.

EXAMPLES XVII.

[Take $\alpha = 1/273$.]

(*The Answers are given on page* 337.)

In Questions 1 to 9 inclusive the air is supposed to contain no moisture.

1. A given mass of air is allowed to expand at constant pressure while its temperature rises from 2° C. to 57° C. Show that its volume will be increased in the ratio of 6 to 5.

2. If a given mass of air is kept at constant volume while its temperature varies—the arrangement in a constant-volume air thermometer—show that the pressure will be proportional to the absolute temperature.

3. If in a constant-volume air thermometer the pressure is that of 30 inches of mercury when the temperature is 10° C., what will the pressure be when the temperature is 25° C.?

4. In a constant-volume air thermometer the pressure is that of 760 millimetres of mercury when the temperature is at the freezing point. At what temperature will the pressure be equal to that of 800 millimetres of mercury?

5. A closed glass tube, filled with air at 0° C. and under atmospheric pressure, is gradually heated. If the tube can safely stand a pressure of 3 atmospheres, to what temperature may it be heated?

6. The volume of air in a room is 2000 cubic feet. Find the weight in grains of the air in the room when the pressure is that of 30 inches of mercury and the temperature is at the freezing point.

Also find the weight of air which *leaves* the room when the pressure falls to 29·25 inches of mercury and the temperature rises to 25° C.

7. The interior of a building contains 100,000 cubic feet of air. Find the weight in pounds of the air in the building at the following pressures and temperatures:—

(i)	pressure,	29·5	inches of mercury;	temperature,	5° C.
(ii)	„	30	„	„	25° C.
(iii)	„	30·5	„	„	35° C.

8. A uniform tube, closed at both ends, contains air, which is separated into two portions by a very small quantity of mercury. The tube is placed in a horizontal position, and the mercury remains at the middle point of the tube when the temperatures of the two portions of air are the same as that of the outside air, which is 15° C. At what point of the tube will the mercury remain at rest when the temperature of one portion is raised to 25° C., the temperature of the other portion being unchanged?

9. The Montgolfier fire-balloon is constructed of paper, and filled with hot air. Prove that in order that the balloon may rise, the air inside must be raised to the temperature

$$(tw + 273\ W)/(w - W) \text{ degrees Centigrade,}$$

where t° C. is the temperature of the surrounding air, W the weight in air of the balloon and car, and w the weight of air which fills the balloon at t° C.

10. Find the weight in grains of 10 cubic feet of air half-saturated with moisture at a pressure of 29·5 inches of mercury and a temperature of 15° C.; being given that the maximum pressure of aqueous vapour at 15° C. is that of ·51 of an inch of mercury.

11. Find the weight in grammes of 1000 litres of atmospheric air at the temperature of 25° C. and the pressure of 750 millimetres of mercury, the dew-point being 10° C.; being given that the maximum pressure of aqueous vapour at 10° C. is that of 9·2 millimetres of mercury.

12. Find the weight in grammes of 15 litres of moist air at a pressure of 748 millimetres of mercury and a temperature of 31° C., the dew-point being 28° C.; being given that the maximum pressure of aqueous vapour at 28° C. is that of 28·1 millimetres of mercury.

What would be the weight of the air if the temperature were raised to 35° C., the pressure remaining unaltered?

13. A closed vessel contains water, and, above the water, a mixture of air and steam, the temperature of the mixture being 120° C., and the pressure gauge standing at 90 inches of mercury. What pressure would be indicated by the gauge if, without change of temperature, the volume of the mixture were reduced by a half?

[The maximum pressure of steam at 120° C. is that of 60·5 inches of mercury.]

CHAPTER XIV.—VARIATION OF ATMOSPHERIC PRESSURE.

148. Variation of Barometric Height.

The height of the barometer is found to vary from place to place at the same time, and from time to time at the same place.

At many meteorological stations a continuous record of the height of the barometer is obtained by means of automatic apparatus, by which curves are drawn on paper showing the atmospheric pressure at any time. In these curves the ordinates represent heights of the barometer, and the abscissæ represent intervals of time.

Changes in the weather are usually preceded by changes in the atmospheric pressure, and consequently by changes in the height of the barometer. On this fact depends the usefulness of the barometer as a weather-glass.

149. Variation of Atmospheric Pressure due to Change of Level.

We have seen (Chap. xi.) that the fall which is observed to take place in the height of a barometer carried up a mountain, is in accordance with hydrostatic principles. It follows from hydrostatics that in a still atmosphere the pressure per square inch at any station must exceed the pressure per square inch at any higher station by the weight of a column of air, one square inch in section, extending from the level of the lower to the level of the higher station.

If the atmosphere were of uniform density, the weights of a given length of a column of air of given sectional area would be the same at all elevations. In that case the fall of the barometer in ascending through any height would be proportional to the height, just as the increase of pressure in descending through any depth in a liquid is proportional to the depth.

Since air is compressible, it follows that the atmosphere cannot be of uniform density. For the density must be greatest at the sea-level, where the pressure is greatest, and must become less the higher we ascend above the sea-level. Hence the weight of a given length of a column of air of given sectional area is less the higher we ascend, and therefore for a *given difference of levels* the fall of the barometer is less the higher we ascend.

For small heights the error committed in supposing that the density remains constant is not very great, so that for small heights the fall of the barometer is approximately proportional to the height ascended. When the height does not exceed a few hundred feet above the sea-level, the fall of the barometer may, for rough purposes, be taken to be 1/9 of an inch per 100 feet.

Ex. 1.—Calculate how much a barometer would fall on carrying it to the top of a tower, 100 feet high, taking the density of air to be uniform for that height.

Specific gravity of mercury is 13·6, and of air is ·001293. The question here is—What is the height of a column of mercury which would produce a pressure equal to the weight of the column of air extending from the bottom to the top of the tower?

Let h denote this height in inches; then

$$h \times 13{\cdot}6 = 100 \times 12 \times {\cdot}001293;$$

from which
$$h = 100 \times 12 \times {\cdot}001293/13{\cdot}6,$$
$$= {\cdot}114 \text{ inch.}$$

Hence the fall would be about 11/100 of an inch.

Ex. 2.—A barometer taken down a diving-bell, is found at a certain depth to stand 18 inches higher than it did above the surface. What is the depth?

Let z denote the depth of water. Then a column of length z of water produces the same pressure as a column of 18 inches of mercury.

Hence
$$z = 18 \times 13{\cdot}6 \text{ inches}$$
$$= 20{\cdot}4 \text{ feet.}$$

The Homogeneous Atmosphere, Arts. 150 to 153.

150. Height of the Homogeneous Atmosphere.

For the purpose of simplifying some formulæ in physics, the conception is made of an atmosphere uniform throughout, and of the same density as the air at a given place and a given time. The length of the column of this imaginary atmosphere which would produce a pressure equal to the atmospheric pressure, is called the *height of the homogeneous atmosphere* for the given place and the given time.

151. Formula for the Height of the Homogeneous Atmosphere.

Let p represent *in absolute measure* the pressure of the atmosphere at a given place and a given time. To fix our ideas we shall suppose that p is expressed in poundals per square foot. Let ρ represent the density in pounds per cubic foot of the air at the given place and time; and let g denote the acceleration of gravity in foot-second units at the given place.

Let H represent in feet the height of the homogeneous atmosphere; then p is equal to the weight in poundals of a column of air of uniform density ρ, one square foot in section, extending to the height H. The volume of this column of air is H cubic feet, its mass is ρH pounds, and its weight is $g\rho H$ poundals.

Hence

$$g\rho H = p;$$

from which $$H = p/g\rho\ldots\ldots\ldots(1)$$

Now let h represent in feet the height of the mercurial barometer at the given place and time, and let σ denote the density of mercury.

Then $$p = g\sigma h.$$

Substituting this value of p in equation (1), we obtain the formula—

$$H = h\sigma/\rho\ldots\ldots\ldots(2)$$

which expresses H in terms of the height of the mercurial barometer and the ratio of the density of mercury to the density of air.

It can be proved from formula (1) that *for a given value of g and a given temperature the height of the homogeneous atmosphere is constant for all atmospheric pressures.* The formula (1) shows that, when g is constant, the value of H varies as p/ρ. But we know from Boyle's Law, that, when the temperature is constant, the pressure of air is proportional to its density. Hence p/ρ is a constant quantity, and therefore, under the supposed conditions, the value of H is constant for all atmospheric pressures, that is, is independent of the height of the barometer.

Formula (2) may be used to calculate the height of the homogeneous atmosphere at a given place and a given temperature. Since, under the supposed conditions, the height of the homogeneous atmosphere is independent of the height of the barometer, we may put for h in formula (2) any value whatever if we put for ρ the density of the air at the given temperature and at the pressure measured by the value taken for h. Also, since the formula involves only the ratio of the density of mercury to the density of air, we may express these densities in any system of units.

Ex.—When the atmospheric pressure at a certain place is that of 29·92 inches of mercury, and the temperature is that of the freezing point, the

mass of a cubic centimetre of air is $\cdot 001293$ of a gramme. Taking the mass of a cubic centimetre of mercury to be $13\cdot 596$ grammes, calculate the height of the homogeneous atmosphere.

Here $h = 29\cdot 92$ inches, $= 2\cdot 4933$ feet,

and $\sigma/\rho = 13\cdot 596/\cdot 001293$;

therefore $H = 2\cdot 4933 \times 13\cdot 596/\cdot 001293$,

$= 26217$ feet.

152. Variation of H due to Variation of Temperature.

Let H_0 denote *for a given place* the height of the homogeneous atmosphere at temperature $0°$ C., and H_t the height at temperature $t°$ C. Then if a denote the coefficient of expansion for air, it can be shown from the laws of gases that

$$H_t = H_0\ (1 + at).$$

To prove this, let p, ρ denote the pressure and density of the atmosphere respectively at any time when the temperature is $t°$ C., and p_0, ρ_0 the values of the corresponding quantities at any time when the temperature is $0°$ C., *the units being the same as in the preceding Article.*

The *volume* of a mass ρ_0 of air at the pressure p_0 and zero temperature is by supposition one cubic foot, and, by Art. 138, the volume in cubic feet of the *same mass* of air at pressure p and temperature $t°$ is $p_0\ (1 + at)/p$. Hence the *mass* of one cubic foot of air at pressure p and temperature $t°$ is $\rho_0 p/p_0\ (1 + at)$. But this is represented by ρ. Hence

$$\rho = \rho_0 p/p_0(1 + at),$$

from which

$$p/\rho = p_0(1 + at)/\rho_0.$$

Now, by formula (1) of the preceding Article,

$$H_0 = p_0/g\rho_0 \text{ and } H_t = p/g\rho.$$

Therefore

$$H_t/H_0 = \frac{p}{\rho}\Big/\frac{p_0}{\rho_0},\ = \frac{p_0}{\rho_0}(1 + at)\Big/\frac{p_0}{\rho_0},$$

$$= 1 + at,$$

or

$$H_t = H_0(1 + at).$$

153. Variation of H due to Variation of Gravity.

For a *given temperature* the value of H will be different at two places at which the values of g, the acceleration of gravity, are different. It follows immediately from formula (1) of Art. 151 that, when the temperature is constant, gH is constant.

For by that formula gH is equal to p/ρ, and p/ρ is constant when the temperature is constant.

Hence if H and H' denote the heights of the homogeneous atmosphere at two stations at which the accelerations of gravity are g and g' respectively, then

$$g'H' = gH,$$

or

$$H'/H = g/g'.$$

Hence *when the temperature is constant, the height of the homogeneous atmosphere is inversely proportional to the acceleration of gravity.*

For a *given temperature* the density of air depends on the pressure. Now, due to the variation of gravity, the same height of the barometer does not represent the same pressure at all stations on the earth's surface. Thus. for example, 30 inches of mercury at Greenwich represent a greater pressure than 30 inches of mercury at Paris, since the weight of a given mass at Greenwich is greater than the weight at Paris. It follows that the density of air is not the same at all stations for the same temperature and the same height of the barometer. Hence in using the formula (2) of Art. 151 to determine the height of the homogeneous atmosphere at a given station from an observed height, h, of the barometer, we must be careful to use for ρ the value of the density of air at the pressure which is represented by the height h of mercury at the place of observation.

Ex.—The height of the homogeneous atmosphere at freezing temperature, at the sea-level, at a station where $g = 32{\cdot}19$ ft./sec.^{2} is 26217. What is the height at a station where $g = 32{\cdot}088$ at the temperature of 15° C.?

At the latter station the height of the homogeneous atmosphere at zero temperature

$$= 26217 \times 32{\cdot}19/32{\cdot}088 \text{ feet};$$

and the height at the temperature 15° C. is

$$= 26217 \times 32{\cdot}19 \times (1 + 15 \times {\cdot}003665)/32{\cdot}088,$$
$$= 27746 \text{ feet.}$$

154. **Determination of Heights by the Barometer.** Let h denote the height of the barometer at any station, and h' the height of the barometer at the same time at any higher station, and let z denote the difference of levels of the two stations. If we assume that the air between the two stations is still, that its temperature is uniform, and that variations of gravity are neglected, the following formula, a proof of which is given below, will give z in terms of h, h', and H, the height of the homogeneous atmosphere for the temperature of the air:—

$$z = H(\text{hyp. log. } h - \text{hyp. log. } h') \dots\dots\dots\dots (1)$$

If logarithms to base 10 are to be used, we must multiply the right-hand side by hyp. log. 10, the value of which is 2·30259. Thus we obtain—

$$z = 2{\cdot}30259H(\log._{10}h - \log._{10}h') \dots\dots\dots\dots (2)$$

For the freezing temperature, and when $g = 32{\cdot}19$, the value of H is 26217 feet (Ex., Art. 151), and the value of $2{\cdot}30259H$ is 60367 feet. *Under these conditions*, therefore, we have

$$z = 60367(\log._{10}h - \log._{10}h') \text{ feet.}$$

If the temperature is t° C., and if the acceleration of gravity at either station is g, we must put instead of the number 60367 the expression

$$60367 \times 32{\cdot}19 \times (1 + at)/g.$$

Hence, for any temperature t° C., and for a given value of g, difference of levels in feet is given by the formula—

$$z = 60367 \times 32{\cdot}19(1 + at)(\log._{10}h - \log._{10}h')/g \dots (3)$$

The following is a proof of formula (1):—

Let AB be the vertical height between the two stations, so that $AB = z$. Let AB be divided into a very large number, n, of equal parts at the points M_1, M_2, M_3, $M_4 \dots$, so that each of the parts AM_1, M_1M_2, M_2M_3, $M_3M_4 \dots$ is equal to z/n, a very small length.

The atmospheric pressure at A, a point on the level of the upper station, is measured by h', the height of the barometer at that station; and similarly the atmospheric pressure at B is measured by h. Let h_1, h_2, h_3, $h_4 \dots$

denote the heights of the barometer at the points M_1, M_2, M_3, M_4... respectively.

Consider the increase of pressure in descending from A through the very small height AM_1. The density of the air between the two points A and

M_1 may, since AM_1 is very small, be taken to be uniform and equal to the density at A. But the pressure at A is that due to a height H of air of the same density as the air at A. Hence the pressure at M_1 may be taken to be that due to a height of the same air equal to (H + height AM_1), that is, to a height $(H+z/n)$. Therefore

$$h_1 : h' = H + z/n : H,$$

or

$$h_1 = h'(1 + z/nH).$$

Similarly the increase of pressure in descending from M_1 to M_2 bears to the actual pressure at M_1 the ratio which M_1M_2 bears to the height of the homogeneous atmosphere at M_1. But, by Art. 151, the height of the homogeneous atmosphere at M_1 is equal to the height of the homogeneous atmosphere at A. Hence h_2 bears to h_1 the same ratio which h_1 bears to h'.

Hence
$$\begin{aligned} h_2 &= h_1(1 + z/nH), \\ &= h'(1 + z/nH)(1 + z/nH), \\ &= h'(1 + z/nH)^2. \end{aligned}$$

Similarly
$$\begin{aligned} h_3 &= h_2(1 + z/nH), \\ &= h'(1 + z/nH)^3, \\ h_4 &= h'(1 + z/nH)^4, \end{aligned}$$

and so on. Proceeding in this way we get

$$h = h'(1 + z/nH)^n.$$

Now the larger the value of n the more nearly is it true that the density of air in each of the strata AM_1, M_1M_2... is uniform, and the more nearly will the last-written equation be exact. If, therefore, we put $n = \infty$ in this equation, we shall get the *exact* equation connecting h, h', z and H. It is known from Algebra that when $n = \infty$, the expression $(1 + z/nH)^n$ becomes equal to $e^{z/H}$, where e denotes the base of the hyperbolic or Napierian system of logarithms.

Hence

$$e^{z/H} = h/h',$$

from which, on equating the hyperbolic logarithms of the two sides, we get

$$z = H(\text{hyp. log. } h - \text{hyp. log. } h');$$

and this is formula (1), from which the formulæ (2) and (3) are derived.

Ex.—At 9 a.m. on June 9, 1893, at a station at the summit of Ben Nevis, the barometer stood at 25·862 inches, the reading of the thermometer being 48°·4 F. At the base station at the same time the barometer

stood at 30·349 inches, the reading of the thermometer being 58°·8 F. Find the difference of levels of the two stations, being given that

$$\log_{10}30·349 = 1·4821444,$$
$$\log_{10}25·862 = 1·4126621,$$

and taking $g = 32·2$ for the latitude of Ben Nevis.

Here we may take the temperature of the air between the two stations to be the arithmetic mean of the temperatures at the two stations. This mean is $\frac{1}{2}(48°·4 + 58°·8)$, or 53°·6 F., that is, 12° C. Hence formula (3) gives in this case

$$z = \frac{60367 \times 32·19 \times (1 + 12 \times ·003665)}{32·2} \times (\log_{10}30·349 - \log_{10}25·862),$$
$$= 63002 \times ·0694823,$$
$$= 4377 \text{ feet.}$$

The difference of levels of the two stations has been accurately determined by trigonometrical observations, and found to be 4365 feet. Hence in this case formula (3) gives a result which is 12 feet in excess of the true value, an error of about 1/4 per cent of the true value.

EXAMPLES.

1. Assuming that $H = 26217$ feet for zero temperature where $g = 32·19$, find the value when

(i) $g = 32·09$, and temperature is 35° C.
(ii) $g = 32·19$, and temperature is $-10°$ C.
(iii) $g = 32·24$, and temperature is $-20°$ C.

Ans. In feet—(i) 29672. (ii) 25256. (iii) 24258

2. Assuming that for average temperatures

$$z = 65000\,(\log_{10}h - \log_{10}h'),$$

find what must be the depth of a mine when the height of the barometer at the bottom is to the height at the top as 5 : 4; given $\log_{10} 2 = ·3010300$.

Ans. 6299 feet.

3. The atmospheric pressure at the top of a mountain is 2/3 of the pressure at the same time at the base. Using the formula in the preceding Example, find the height of the mountain, given $\log_{10} 2$ and $\log_{10} 3 = ·4771213$.

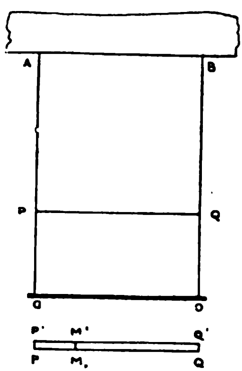
Ans. 11446 feet.

CHAPTER XV.

SECTION I.—TENSION OF SURFACES UNDER FLUID PRESSURE.

155. Measure of the Tension of a Surface.

ABCD is a rectangular sheet of some thin material of uniform thickness, suspended from a fixed beam with the edges AD and BC vertical. CD is a uniform rod suspended from the lower horizontal edge of the sheet. We may suppose that ABCD is a wall-map, and that CD is the roller attached to the end of the map.



Let l denote the breadth CD of the sheet, k the thickness of the sheet, and W the weight of the rod CD.

Leaving the weight of the sheet out of account, let us consider the stress produced in the sheet by the weight W. This stress will be a tension tending to tear the sheet asunder.

Take a section of the sheet perpendicular to the plane of the sheet through a horizontal line PQ in the sheet. Since the sheet is supposed to be thin, this section will be a very narrow rectangle PQQ′P′, whose length PQ is l, whose breadth QQ′ is k, and whose area is kl units of area.

The stress across the section PQQ′P′ will consist of two equal and opposite tensions, W, W, perpendicular to the section, tending to tear the lower part of the sheet from the upper. The intensity of this stress, that is, the amount of the force per unit of area, is W/kl. Call this s; then

$$s = W/kl.$$

It is convenient to speak of the stress across the section PQQ′P′ as the *tension of the sheet across the line PQ*. From PQ cut off PM equal to the unit of length, and draw MM′ parallel to PP′. Then the area of PM′ is $1 \times k$, that is, k units of area.

Let the stress on this area be denoted by t; then t is called *the tension of the sheet per unit of length across the line PQ*.

It follows that

$$t = W/l, = k \,.\, (W/kl),$$
$$= ks.$$

In the case just considered a tension is caused in a thin plane sheet by the weight of a rod. Where fluids are contained in vessels whose sides are made of thin material, tensions will be produced in the sides by the pressures of the fluids, and will be measured in the same way as the tension of a plane sheet.

156. Greatest Tension a Surface of given Material and given Thickness can bear.

If a rod of any material is suspended from one end, and a weight hung at the other end, a stress of the nature of a tension will be produced in the rod tending to pull the rod asunder. It is found by experiment that for each kind of material there is a certain intensity of stress which is sufficient to fracture the rod. This is called the *tearing stress* for the material of which the rod is composed, and the magnitude of this stress is a measure of the *tensile strength* or *tenacity* of the material.

Thus, for example, when we say that the tenacity of copper is 40000 lbwt. per square inch, we mean that a rod of copper, one square inch in section, will be pulled asunder by a tension greater than 40000 lbwt.; a rod of copper, half a square inch in section, by a tension greater than 20000 lbwt.; a rod of copper, 4 square inches in section, by a tension greater than 160000 lbwt.; and so on.

If S denotes the tenacity of a given substance, and T the greatest tension per unit of length which a sheet of the substance of thickness k can bear, then

$$T = k\,S.$$

157. Tension of a Cylinder under Internal Fluid Pressure.—Let a closed hollow cylinder, whose sides are made of some thin material, contain a fluid *whose weight is*

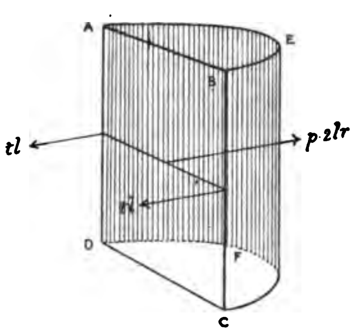
neglected. Then the pressure of the fluid will be the same at all points of the sides.

Consider any element of the curved surface of the cylinder. Due to the pressure of the fluid on the element, tensions will be produced between the element and the surrounding parts of the surface. Since the pressure on every element of the surface is perpendicular to the axis there will be no tension on the element in the direction of the axis. Also, it is evident that the tension per unit of length across all generating lines of the surface will be the same.

Let t denote the tension across unit length of a generating line, p the pressure of the fluid, and r the radius of the cylinder. Then—

$$t = pr.$$

To prove this formula, consider the equilibrium of that part AEBCFD of the surface of the cylinder, which is cut off by a plane ABCD, drawn through the axis of the cylinder, and by two planes AEB and CFD, drawn perpendicular to the axis of the cylinder at a distance l apart. Then ABCD is a rectangle whose sides are l and $2r$.



The resultant pressure of the fluid on the curved surface AEBCFD is (Art. 76) equal and opposite to the resultant pressure on the rectangle ABCD. The magnitude of this pressure is $p \times$ area of the rectangle, that is, is equal to $p \,.\, 2lr$; and the line of action is the line through the centre of gravity of the rectangle perpendicular to the plane of the rectangle. Hence the resultant pressure of the fluid on the curved surface is a force equal to $p \,.\, 2lr$, acting outwards on the curved surface along the line drawn from the C.G. of the rectangle perpendicular to the plane of the rectangle.

This pressure on the curved surface will produce tensions

across the generating lines BC and AD. These tensions will act along the surface, perpendicular to BC and AD, that is, parallel to the direction of the resultant pressure on the curved surface. Since t represents the tension per unit of length across a generating line, the tensions across BC and AD will be equivalent to forces, each equal to tl, acting at the middle points of BC and AD respectively. It follows that the material of which AEBCFD is composed may be taken to be in equilibrium under the action of three parallel forces $p \,.\, 2lr$, tl, and tl respectively, acting in the plane which is parallel to, and midway between, the planes AEB and CDF, as shown in the diagram. Hence—

$$p \,.\, 2lr = 2tl\,;$$

from which $$t = pr.$$

This investigation evidently holds whether the material of which the sides are composed is, or is not, flexible.

Cor. Let k denote the thickness (supposed to be uniform) of the sides, S the tenacity of the material of which the sides are made, and P the greatest pressure of fluid to which the cylinder may be subjected without bursting. Then the greatest surface tension which the sides can bear is kS, and this is equal to Pr. Hence—

$$Pr = kS\,;$$

from which $$P = kS/r,$$

a formula for P, the *bursting pressure.*

Ex.—A hollow cylinder, 8 feet in diameter, whose sides are made of iron 1/8 of an inch thick, is placed with its axis vertical. Find the greatest depth of mercury which the cylinder can contain without bursting, being given that the tenacity of iron is 56000 lbwt. per square inch, and that a cubic inch of mercury weighs 1/2 of a pound.

For different depths below the surface of a liquid the pressure is proportional to the depth. Hence the intensity of pressure of the mercury on the sides of the cylinder will be greatest at the base of the cylinder; and it is at that place that bursting will take place.

Let h denote the depth in inches of the mercury when the cylinder bursts. The pressure at the base of the cylinder is $h/2$ lbwt. per square inch, and

is equal to the bursting pressure for the cylinder. In the above formula, therefore, for P we put

$$P = h/2,\ k = 1/8,\ S = 56000,\ r = 4 \times 12,$$

and thus obtain the equation—

$$h/2 = (1/8) \times 56000/48,$$

from which $h = 292$ inches nearly.

Hence the greatest depth of mercury the cylinder can hold without bursting is 292 inches or $24\frac{1}{3}$ feet.

158. Tension of a Sphere under Internal Fluid Pressure.

Let a hollow sphere, whose surface is made of some thin material, contain *weightless* fluid. Then, as in the case of the cylinder, the pressure of the fluid will produce tensions in the surface of the sphere. It is evident that the tension across an infinitesimal line of given length, drawn on the surface of the sphere, will be the same at all points.

Let p denote the pressure of the fluid on the surface of the sphere, r the radius of the sphere. Let t be the measure of the surface tension; then the tension across any infinitesimal length drawn on the surface of the sphere is *at the rate of* t units of force per unit of length. We shall prove that

$$t = \tfrac{1}{2}pr.$$

Let ABCD (see figure, Art. 76) be the sphere, and let a plane be drawn through G the centre, dividing the surface into two hemispheres. Consider the equilibrium of the hemisphere BAD.

The pressure of the fluid on the hemisphere BAD will produce tensions in the surface across the circle BED, tending to tear the two hemispheres asunder. At every point of this circle there will be a tension along the surface in the direction perpendicular to the plane of the circle, and the resultant of all the tensions across this circle will be

$$= t \times \text{circumference of circle},$$
$$= t\ .\ 2\pi r.$$

Since the part BAD of the surface is in equilibrium, the

resultant pressure of the fluid on this part of the surface is balanced by the tension across the circle BED. Now by Art. 76 the resultant pressure of the fluid on the hemisphere BAD is $p \cdot \pi r^2$.

Hence $$t \cdot 2\pi r = p \cdot \pi r^2,$$
from which $$t = \tfrac{1}{2}pr.$$

Cor. Let k denote the thickness (supposed to be uniform) of the spherical shell, S the tenacity of the material of which the shell is made, and P the bursting pressure. The greatest tension which the surface of the sphere can bear without tearing is kS, and this must be equal to $\frac{1}{2}Pr$.

Hence $$\tfrac{1}{2}Pr = kS;$$
from which $$P = 2kS/r.$$

By comparing this formula with the corresponding formula for a cylinder (Cor., Art. 157), we see that the bursting pressure for a spherical shell is double the bursting pressure for a cylinder, whose sides are made of the same material and of the same thickness as the spherical shell, and the radius of whose section is equal to the radius of the shell.

We have supposed in this and the preceding article that there is no external pressure. If there is external pressure, it may be taken into account by supposing that p represents the excess of the internal over the external pressure.

Ex.—The tenacity of copper being given as 40000 lbwt. per square inch, find the bursting pressure, in lbwt. per square inch, of a hollow copper globe 2 feet in diameter and a quarter of an inch thick.

Here $$k = 1/4,\ r = 12,\ S = 40000,$$
and therefore $$P = 2kS/r, = 2 \times \tfrac{1}{4} \times 40000/12, = 1667 \text{ nearly}.$$

The bursting pressure is therefore 1667 lbwt. per square inch.

EXAMPLES XVIII.

(*The Answers are given on page* 337.)

1. Taking the tenacity of copper to be 40000 lbwt. per square inch, find—

(i) The bursting pressure of a copper globe, 18 inches in diameter, and 1/10 of an inch thick.

(ii) The least thickness of the sides of a copper cylinder which is to bear

an internal pressure of 10 atmospheres, taking an atmosphere to be a pressure of 15 lbwt. per square inch, when the radius is 1 foot.

(iii) The greatest radius of a spherical copper boiler, 1/4 of an inch thick, which will bear a pressure of 1000 lbwt. per square inch.

2. Find the tenacity of cast-iron, in lbwt. per square inch, being given that the bursting pressure of a cast-iron pipe, 2 feet in diameter and 1/6 of an inch thick, is 250 lbwt. per square inch.

3. A water hose, whose internal diameter is $2\frac{3}{4}$ inches, is just able to bear a pressure of 100 lbwt. per square inch. If the thickness of the hose is 1/8 of an inch, calculate the tenacity of the material of which the hose is made.

4. A closed cylinder, whose sides are made of thin material, is full of liquid, the weight of unit volume of which is w. The cylinder is hinged along a generating line to a fixed horizontal beam. Show that if t denotes the tension per unit of length, due to the weight of the liquid, across either of the generating lines in which the surface is cut by a horizontal plane through the axis, and r denotes the radius of the section of the cylinder.

$$t = r^2 w\ (1 + \pi/4).$$

5. A spherical shell, radius r, made of thin material, is full of liquid, the weight of unit volume of which is w. The shell is freely suspended from the highest point. Show that if t denotes the tension per unit of length, due to the weight of the liquid, across the circle in which the surface is cut by a horizontal plane through the centre,

$$t = 5r^2 w/6.$$

6. A hemispherical vessel, radius r, made of thin material, is fastened along the rim to a fixed horizontal ring of the same radius as the shell. The vessel is filled with liquid, the weight of unit volume of which is w. Show that if the tension on the ring per unit of length is denoted by t, then

$$t = r^2 w/3.$$

SECTION II.—SURFACE TENSION OF LIQUIDS. CAPILLARITY.

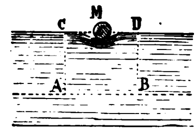
159. Surface Tension of Liquids.

It is inferred from experiment* (i) that there is a force of the nature of a surface tension in the free surface of every liquid, (ii) that this surface tension is different in different liquids.

The phenomena of drops and soap-bubbles point to the

* For a full account of experiments which show the existence of surface tension in liquids the student may be referred to *Soap Bubbles*, by C. V. Boys, F.R.S., published in the Romance of Science Series by the Society for Promoting Christian Knowledge.

conclusion that the free surface of a liquid acts like an elastic surface which is in a state of tension. To the same conclusion we are led by such experiments as that of floating a needle on the surface of water. If the needle is placed very gently on the water, it will float without breaking through the surface, its weight being supported by the tension of the surface. It will be observed that the needle lies in a distinct depression of the surface, as shown in the accompanying figure, in which CD is a section of the surface of the water by a plane perpendicular to the needle M.



That the surface tension is different in different liquids may be illustrated by the experiment of letting a drop of alcohol fall on the surface of a very thin layer of coloured water on a plate. It is observed that the alcohol is drawn by the water in all directions, with the result that a small spot on the plate becomes bare. This shows that the surface tension of water is greater than that of alcohol.

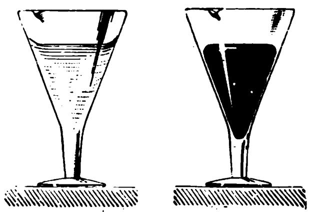
The surface tension of a liquid is measured in the same way as the tension of a thin surface (Art. 155), that is, in terms of force per unit of length. The amount is small in all liquids. In water, alcohol, and mercury the surface tension is about 3, 1, and 19 grains weight per inch respectively.

160. Capillary Phenomena.

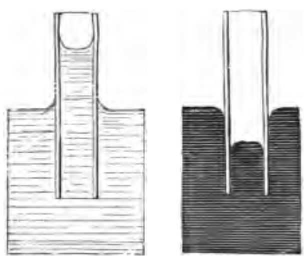
When a body is partly immersed in a liquid, it is observed that the surface of the liquid at the points where it meets the body is either elevated above or depressed below the general surface of the liquid. If the liquid wets the body, the surface of the liquid at the points of contact with the body is raised; but if the liquid does not wet the body, the surface of the liquid at the points of contact is depressed.

When a liquid is contained in a vessel, the surface of the liquid at the points of contact with the sides of the vessel is elevated or depressed according as the liquid does or does not wet the sides of the vessel. The figure shows on the left a glass vessel containing water, which wets glass, and on the

right a glass vessel containing mercury, which does not wet glass. In the case of the water the surface is raised at the sides of the vessel, and appears concave, while in the case of the mercury the surface is depressed at the sides of the vessel, and appears convex.



When a glass tube of small bore (usually spoken of as a capillary tube) is dipped into a liquid, the surface of the liquid inside the tube stands higher or lower than the general surface of the liquid outside according as the liquid does or does not wet the sides of the tube. The figure shows on the left the elevation of the surface of water inside a glass tube, and on the right the depression of the surface of mercury, the surface being concave in the case of the water and convex in the case of the mercury. For a tube 1/25 of an inch in diameter the elevation of the surface of the water is about $1\frac{1}{5}$ inches, and for a tube 2/25 of an inch in diameter the depression of the surface of the mercury is about 1/5 of an inch.



It is found by experiment that with tubes of given material, but of different diameters, dipped into a given liquid, the elevation or depression varies inversely as the diameter. This experimental result, which is known as the *law of diameters*, can be shown to follow from the assumption of the existence of a surface tension in liquids. (See Art. 162.)

When two plates are immersed in a liquid, so that their planes are parallel and separated by a small distance, an elevation or a depression of the surface of the liquid between the two plates is observed. As in the case of tubes, the surface is elevated if the liquid wet the plates, and is de-

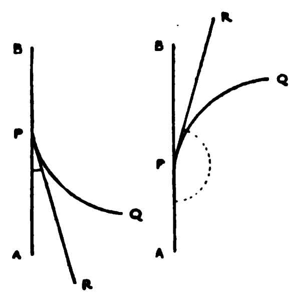
pressed if the liquid does not wet the plates. With given materials, it is found by experiment that the elevation or depression of the surface of the liquid between two parallel plates is equal to the elevation or depression in a tube, whose radius is equal to the distance between the plates. It will be shown in Art. 162 that this result also follows from the theory of surface tension.

The elevations and depressions of the surface of liquids in narrow tubes and between plates at a small distance apart are classed together under the name of *capillary phenomena.* It is usual to say that the phenomena are due to *capillary action* or *capillarity.*

There are many illustrations of capillary action with which every one is familiar, such as the absorption of water by a sponge, the ascent of the oil in the wick of a lamp, the transference of water from one vessel to another through a woollen thread acting like a siphon, &c.

161. Angle of Contact of a Liquid with a Surface.

It is found by experiment that when the surface of a solid of given material is in contact with a given liquid, the direction of the surface of the liquid at the points where it meets the solid is inclined at a definite angle to the surface of the solid. The angle of contact varies with the nature of the surface of the solid and with the nature of the liquid.



If we measure the angle of contact from the surface of the solid below the surface of the liquid to the surface of the liquid, then in the case where the liquid is elevated by contact with the surface of the solid, the angle of contact is an acute angle, and in the case where the liquid is depressed the angle of contact is an obtuse angle. The two cases are illustrated in the accompanying figures. In the figures AB represents the surface of the solid, PQ the surface of the liquid, and PR the direction of the surface of the liquid at a point P

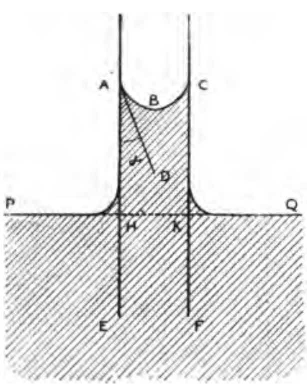
where it meets the surface of the solid. In the figure on the left, which may be taken to represent the contact of water and glass, the surface of the liquid is concave, and the angle of contact, APR, is acute. In the figure on the right, which may be taken to represent the contact of mercury and glass, the surface of the liquid is convex, and the angle of contact, APR, is an obtuse angle.

162. **Formulæ for the Elevation of Liquids in Capillary Tubes and between Plates.**

Formulæ may be deduced from the supposition of surface tension in liquids for the elevation (or depression) of liquids in capillary tubes and between plates. We shall suppose that the surface of the liquid is elevated, as in the case of water in contact with glass. The working out of the case in which the liquid is depressed may be left as an exercise for the student.

I. **Rise of Liquid in a Capillary Tube.**

Let AEFC represent a section of a capillary tube, dipped into a liquid which wets the sides of the tube. The surface, ABC, of the liquid inside the tube will stand higher than the level of the surface, PQ, of the liquid outside the tube. Let AD be the direction of the surface of the liquid at the point A, so that the angle HAD is the angle of contact of the liquid and the tube. For a given liquid, and for tubes of given material, this angle is constant, and may be determined by experiment. Let it be denoted by a.



Let r denote the radius of the tube, h the average height of ABC above PHKQ, the level of the liquid outside the tube, and w the weight of unit volume of the liquid. Let t denote the surface tension of the liquid, measured as a force per unit of length.

Consider the equilibrium of the liquid in AHKC. This liquid is acted on by four forces (in addition to the normal pressures of the sides of the tube, the resultant of which is zero)—(i) the atmospheric pressure on the surface ABC; (ii) the upward pressure of the liquid across the horizontal section through HK; (iii) the weight of the liquid; (iv) the surface tension at circle (represented by the points A and C in the figure) in which the surface ABC meets the sides of the tube. Now since HK is at the level of PQ, the pressure at points in the horizontal section through HK is equal to the atmospheric pressure. It follows that the upward pressure of the liquid on AHKC is equal to the downward pressure of the atmosphere on the surface ABC of the liquid. Hence the forces (i) and (ii) balance each other, and therefore the forces (iii) and (iv) also balance each other.

The force (iii) is the weight of the column AHKC of liquid, which may be taken to be a cylinder whose radius is r and height h. The volume of this cylinder is $\pi r^2 h$, and therefore the weight of the column of liquid may be taken to be $\pi r^2 hw$.

The force (iv) is the force which the tube exerts on the liquid at the circle in which the surface ABC meets the sides of the tube. [It may help the student to see the action of this force by imagining that ABC is a thin elastic surface attached to a ring fixed to the sides of the tube.] Since t is the force per unit of length of this circle, the whole force is (as in Art. 158) equal to $2\pi rt$. This force acts at every point of the circle in the direction of the surface of the liquid at the points of contact, that is, acts in the direction inclined to the vertical at an angle a. The vertical component of this force is (Art. 30) $2\pi rt \cos a$.

Now since the forces (iii) and (iv) balance, and since the force (iii) is a vertical force, it follows that the force (iii) must be in equilibrium with the vertical component of the force (iv)

Hence we arrive at the equation—

$$\pi r^2 hw = 2\pi rt \cos a,$$

from which $$h = 2t \cos a / rw \ldots\ldots\ldots\ldots(1)$$

For tubes of given material, but of different diameters, and

for a given liquid, t, w, and a are all constant, and therefore rh is constant, or h varies inversely as r. Hence the elevation of a given liquid in tubes of given material varies inversely as the radius, the result known as the law of diameters (Art. 160).

II. Rise of Liquid between Parallel Plates.

Take the same figure to represent a section made by a vertical plane drawn perpendicular to the planes of two parallel plates, at a small distance apart, which are partly immersed in a vertical position in a liquid which wets their surfaces. The surface of the liquid between the plates will stand at a higher level than the general surface of the liquid.

Let AE and CF in the figure represent the plates, ABC the surface of the liquid between the plates, and PHKQ the level of the surface outside. Let h denote the average height to which the liquid rises between the plates, and d the distance between the plates. Also let t denote the surface tension of the liquid, a the angle HAD, the angle of contact of the liquid with the surfaces of the plates, and w the weight of unit volume of the liquid.

Consider the equilibrium of that part of the liquid between the two plates which is contained between two planes parallel to the plane of the section represented by the figure, and at unit distance apart. As in the case of the liquid in a capillary tube, the weight of the liquid between these two planes and the two plates will be supported by the vertical components of the surface tensions at the edges of the surface of the liquid. The line of contact of this liquid with each of the plates will be a straight line of unit length, and the vertical component of the resultant force of the plates on the liquid will be $2t \cos a$. This must be equal to the weight of the liquid under consideration, that is, equal to $h \times d \times 1 \times w$.

Hence $$h\,d\,w = 2t \cos a,$$
from which $$h = 2t \cos a / d\,w \qquad (2)$$

Hence, with given materials, the rise of liquid between two parallel plates is inversely proportional to the distance between the plates.

By comparing the formulæ (1) and (2) we see that, with given materials, the rise of liquid in a circular tube is equal to the rise between two parallel plates, whose distance apart is equal to the radius of the tube. For, with given materials, t, w, and a are constant, and formulæ (1) and (2) will therefore give the same value for h if r is equal to d.

III Rise of Liquid between two Plates inclined at an Angle.

When the plates, instead of being parallel, are inclined to each other like two folds of a half-opened draught-screen, the liquid between the plates will not rise to a uniform height. It follows from formula (2) that the liquid will stand highest close to the line of intersection of the plates, where the distance between the plates is small, and that the rise of the liquid will be less and less at greater and greater distances from the line of intersection. For it follows from the formula that the rise at any distance from the line of intersection is inversely proportional to the distance between the plates, this distance being measured in a plane perpendicular to the plane bisecting the angle between the plates. But the distance so measured is proportional to the distance of the line in which this plane meets the plane of either of the plates from the line of intersection of the plates. Hence it follows that the surface of the liquid meets each of the plates in a curve, which is such that the height of any point of the curve above the level of the liquid outside the plates is inversely proportional to the distance of the point from the line of intersection of the two plates. This curve, therefore, is such that the ordinate is inversely proportional to the abscissa, a property which belongs only to the rectangular hyperbola. If in the figure of page 213 we take OY to be the line of intersection of the two plates, and OX to be the line in which the level of the surface of the liquid outside the plates meets one of the plates, then the dotted line A′M′N′...B′, which represents part of a rectangular hyperbola, may be taken to be part of the line in which the surface of the liquid inside the plates meets one of the plates.

CHAPTER XVI.—FLOW OF LIQUIDS THROUGH ORIFICES.

163. Torricelli's Theorem.

The following theorem was stated by Torricelli in 1643 as the result of experiments on the velocity with which particles of water issue from a small orifice in the base or side of an open vessel containing water:—

If a small orifice is made in an open vessel containing water, the particles of the water will issue from the vessel with the velocity that would be acquired by a body in falling in vacuo under gravity through a height equal to the height of the surface of the water in the vessel above the orifice.

To express this result in symbols, let h denote the height of the free surface of the liquid above the orifice (whose linear dimensions are supposed to be very small compared with h), g the acceleration of gravity, and v the velocity of the issuing particles;

then $$v = \sqrt{2gh}.$$

The student will notice that this formula for the velocity of the issuing particles involves only g and h, and therefore makes the velocity to be independent of the position and inclination of the orifice, as long as the depth of the orifice is kept constant.

Accurate experiments, carried out in more recent times, show that Torricelli's Theorem gives a value for the velocity of the issuing particles which is in excess of the true value. The difference between the value $\sqrt{2gh}$, which may be called the *theoretical value*, and the actual value is found to depend on the form and size of the orifice. In the case in which the orifice is very small compared with h, the theoretical value has been found to agree very approximately with the value deduced from experiment.

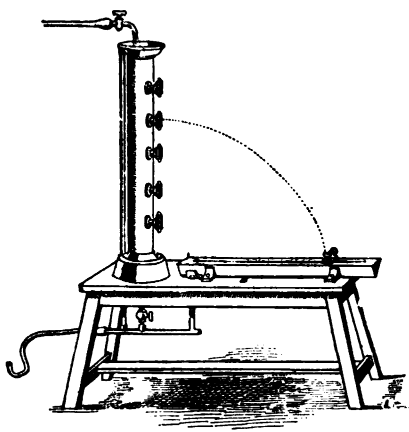
164. Experimental Verification of Torricelli's Theorem.

If a very small orifice is made in the vertical side of a vessel containing water, the particles of the water will be discharged *horizontally* from the orifice. If the surface of the water is maintained at a constant level, the particles of the water will be discharged with constant velocity, and will

proceed, under the action of gravity, to describe in the air the same parabola. The issuing stream will therefore appear as a jet in the form of a parabola.

A formula may be constructed for calculating the velocity of discharge when we know the following lengths, which may be determined by actual measurement:—(i) The distance from the side of the vessel to the point where the jet meets a horizontal plane, and (ii) the height of the orifice above that plane.

The accompanying figure shows an arrangement by means of which the experiment may be carried out. The surface of water in a vessel with vertical sides is maintained at a constant level, and a small orifice is made in the side of the vessel, so that the water is discharged horizontally. The jet enters a horizontal trough at a point whose distance from the side of the vessel is measured on a scale of lengths attached to the trough. From this measured length and the known height of the orifice above the trough, the velocity with which the particles of water leave the vessel may be determined, the resistance of the air being neglected.



Let x denote the horizontal distance of the point where the jet enters the trough from the side of the vessel, y the height of the orifice above the trough, and v the velocity with which the particles of water are discharged, the direction of this velocity being horizontal.

If we neglect the resistance of the air, there will be no horizontal force acting on a particle of the water during its motion from the orifice to the trough, and therefore the particle maintains its horizontal velocity unchanged during the motion. Hence, if t denote the time of the motion, the horizontal distance x is described in time t with constant horizontal velocity v, and therefore

$$x = vt \text{..........} \quad (1)$$

Also in the same time t, the particle falls through the height y under the action of gravity. Hence

$$y = gt^2/2, \text{..........} \quad (2)$$

where g denotes the acceleration of gravity.

From equation (1) we get $t=x/v$, and on substituting this value of t in equation (2), we obtain the equation

$$y=gx^2/2v^2,$$

from which $$v=\sqrt{gx^2/2y}, \qquad\qquad (3)$$

a formula which expresses v in terms of the measured distances x and y.

The degree of exactness of Torricelli's Theorem may be determined by comparing the value of the velocity calculated by this formula with the theoretical value $\sqrt{2gh}$, where h is the depth of the orifice below the surface of the water. By opening orifices at different points of the side of the vessel we vary the experiment by varying x and y, and thus obtain from the above formula values of the velocity for different depths below the surface of the water. Each of these values may be compared with the corresponding theoretical value deduced from Torricelli's Theorem.

165. Proof of Torricelli's Formula from the Equation of Energy.

Let a very small orifice be made in the side or base of a vessel at a depth h feet below the free surface of liquid in the vessel, and let the surface of the liquid be maintained at a constant level, as in the experiment described in Art. 164. Then v, the velocity (in feet per second) of the particles of liquid which issue from the orifice, will be constant.

Since the surface of the liquid is maintained at a level, it follows that when a quantity of liquid has been discharged from the vessel an equal quantity must have entered the vessel. Hence the work done by gravity on the liquid where a mass of m pounds has been discharged is equal to the work that gravity would do on that mass in causing it to fall through the height h. This amount of work is equal to mgh foot-poundals.

Now, if we agree to neglect (i) the work done against resistances, and (ii) the motion of the liquid in the vessel, the work done by gravity on the liquid while the mass m is being discharged may be equated to the kinetic energy gained by that mass (Art. 19). This kinetic energy is equal to $mv^2/2$ foot-poundals.

Hence $$mv^2/2=mgh,$$

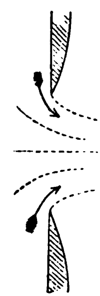
from which $$v=\sqrt{2gh},$$

which is Torricelli's Formula. In obtaining this formula we have neglected the energy lost due to imperfect fluidity, &c.,

and we should expect therefore that the formula will give a value for the velocity which is in excess of the actual value.

166. The Vena Contracta.

When the orifice is not very small, it is found that the stream contracts in breadth for some distance from the orifice. This contraction is well marked in the case in which the orifice is a hole of some size in the side or base of a vessel made of thin material. This case is illustrated in the accompanying figure, in which the dotted lines represent the paths of the particles of water which issue from different parts of the orifice. Particles which are issuing from the centre of the orifice are moving at right angles to the plane of the orifice, while particles which are escaping near the edge of the orifice have a motion inclined at an angle less than a right angle to the plane of the orifice. It follows that the particles near the edge, after passing the orifice, will continue for a short distance to move towards the centre of the stream, with the result that there will be a diminution of the breadth of the stream.



The part of the stream where the contraction ceases is called the *vena contracta*. At this part of the jet the particles are all moving in parallel lines.

The amount of contraction depends on the form of the orifice and the thickness of the side of the vessel. When the thickness is considerable, and the opening presents no sharp edges at the inner surface of the sides, the contraction is small. In the case in which the orifice is a hole in the side of a vessel made of thin material, the area of the section of the *vena contracta* is found to be about 5/8 of the area of the orifice.

167. Quantity of Liquid discharged in Given Time when the Level of the Surface is Constant.

It is easy to frame a formula for the quantity of liquid which is discharged in a given time through an orifice in the base or side of a vessel containing liquid, by making the following suppositions:—(i) That the free surface of the liquid is main-

tained at the same level; and (ii) that all the particles issue from the orifice in a direction perpendicular to the plane of the orifice.

Let v denote the velocity in feet per second of the particles of liquid, and A the area of the orifice in square feet. Then on the supposition that the surface is maintained at constant level, v will be constant, and on the second supposition vA will be the number of cubic feet of liquid which flows through the orifice in one second. Hence if Q denotes the number of cubic feet of liquid which flows through the orifice in t seconds,

$$Q = Avt \qquad (1)$$

The error involved in the second supposition will not be great in the case in which the orifice is very small. And in this case we may, by Torricelli's Theorem, write for v the value $\sqrt{2gh}$, where h is the depth of the orifice in feet below the surface of the liquid, and g is the acceleration of gravity in feet and seconds. Hence we have

$$Q = At\sqrt{2gh} \qquad (2)$$

The supposition that the escaping particles of liquid are all moving perpendicular to the plane of the orifice involves an error the amount of which depends on the form and size of the orifice and the thickness of the sides of the vessel. The formula (2) will in all cases give a value for Q which is in excess of the true value. A more correct result may be obtained by taking A to represent the area of the *vena contracta*, and h the depth of this part of the stream below the surface of the liquid.

In practice engineers use the formula (2), taking A to represent the product of the area of the orifice by a proper fraction, called the *coefficient of discharge*, which is determined by experiment in each case. This product is called the *effective area* of the orifice.*

Ex. 1.—A reservoir is kept full, so that the surface of the water is at a constant height of 12 feet above a small hole, the effective area of whose

* The advanced student may refer to a paper "On the Flow of Water through Orifices," by Professor James Thomson, in the British Association Report for the year 1876, in which the problem of the discharge of water through orifices is subjected to rigid mathematical treatment.

cross section is $\sqrt{3}$ square inches. Find how many cubic feet of water flow out in an hour. ($g=32$.)

We use formula (2), taking a foot as the unit of length, and a second as the unit of time. Then

$$A=\sqrt{3}/144,\ t=60\times 60,\ g=32,\ h=12,$$

and the formula gives

$$Q=(\sqrt{3}/144)\,.\,60\times 60\,.\,\sqrt{2\times 32\times 12},$$
$$=1200 \text{ cubic feet.}$$

Ex. 2.—A cylindrical vessel, with a small hole in the bottom, is 4 feet high. It is kept full of water, and it is found that 4 cubic feet of water flow out of the hole in 3 minutes. What is the size of the hole?

Here, in the same units as in Ex. 1,

$$Q=4,\ t=3\times 60,\ h=4,\ g=32,$$

and A is the unknown. Formula (2) gives

$$4=A\times 3\times 60\sqrt{2\times 32\times 4},$$
$$=A\times 3\times 60\times 16;$$

from which $A=1/(3\times 60\times 4)$ of a square foot,
$$=1/5 \text{ of a square inch.}$$

168. Velocity of the Descending Surface of the Liquid.

If liquid is being discharged from an orifice in a vessel, and if no liquid enters the vessel, the free surface of the liquid will descend. Taking the orifice to be very small, we can construct a formula for the velocity of the descending surface in terms of the following quantities:—(i) The effective area of the orifice, (ii) the area of the section of the vessel at the surface of the liquid, and (iii) the depth of the orifice below the surface.

Let v denote the velocity with which the particles of liquid are discharged from the orifice when the depth of the orifice is h, so that v is equal to $\sqrt{2gh}$. Let v' denote the velocity of the descending surface of the liquid in the vessel at the same instant. Let A and B denote respectively the effective area of the orifice and the area of the section of the vessel at the surface of the liquid.

Let t denote an infinitesimal element of time, so that during the time t the velocities v and v' may be considered to be constant.

In the time t the volume which flows out at the orifice is Avt, and in the same time the surface of the liquid falls through the height $v't$. Now since t is an infinitesimal time, the depth $v't$ is also infinitesimal, and whatever be the shape of the vessel, the sides of the vessel may be considered for that depth to be vertical. Hence the volume of liquid which leaves the vessel in time t must be equal to the product of $v't$ and B. Equating this product to Avt, we get

$$Bv't = Avt,$$

from which

$$v' = Av/B,$$
$$= A\sqrt{2gh}/B, \qquad (1)$$

a formula for the velocity of the descending surface in terms of A, B, and h.

In the case in which the vessel is *a cylinder placed with its axis vertical*, it can be shown that *the velocity of the descending surface will be uniformly retarded.* For B is in that case constant, and if we write a for gA^2/B^2, then a will also be constant. But from the formula for v' we get

$$v'^2 = 2\,.\,(gA^2/B^2)\,.\,h,$$
$$= 2ah.$$

From the formulæ for uniformly accelerated motion we see that v' is numerically equal to the velocity of a body which has moved in a straight line from rest over the distance h with uniform acceleration a. And since the surface is descending, h and v' are decreasing. Hence it follows that the velocity of the descending surface is decreasing uniformly with the time, and that the height through which the surface of liquid falls in a given time may be calculated by the known formulæ for the space described by a body whose motion is uniformly retarded.

Thus the *average velocity of the descending surface* during any time t *is equal to half the sum of the initial and final velocities.* Hence, if h and h' denote the initial and final heights of the surface above the orifice respectively, we get

$$h - h' = \text{average velocity} \times \text{time},$$
$$= \tfrac{1}{2}\left(\frac{A}{B}\sqrt{2gh} + \frac{A}{B}\sqrt{2gh'}\right)t,$$
$$= (\sqrt{h} + \sqrt{h'})\,.\,At\sqrt{g}/B\sqrt{2} \qquad (2)$$

Also the volume of liquid, V say, that flows out of the vessel while the surface falls through the height $h-h'$ is given by

$$V = B\,(h-h'),$$
$$= (\sqrt{h}+\sqrt{h'})\,At\sqrt{g}/\sqrt{2}\ldots\ldots\ldots\ldots(3)$$

Ex.—A cylindrical vessel is 10 feet high, and its base, which is horizontal, has a radius of 1 foot; it is filled with water, and a small hole is made in the bottom. Compare the rate of descent of the surface when the hole is first opened with the rate when the vessel is half empty. If the vessel is half emptied in 20 minutes, what is the effective area of the hole?

By formula (1) the velocity of the descending surface is proportional to the square root of the height of the surface above the orifice. Hence

velocity when the vessel is full : velocity when it is half emptied

$$= \sqrt{10} : \sqrt{5},$$
$$= \sqrt{2:1}, = 7:5 \text{ nearly}.$$

In the second part of the question, take a foot as the unit of length and a second as the unit of time. Then B, the area of the section of the cylinder, is $\pi \times 1^2, = \pi$ square feet; and if A denotes the effective area of the hole, the average velocity of the descending surface during the fall of half the height of the cylinder,

$$= \tfrac{1}{2}\cdot\frac{A}{\pi}(\sqrt{2g\times 10}+\sqrt{2g\times 5})$$

$$= \tfrac{1}{2}\cdot\frac{8A}{\pi}(\sqrt{10}+\sqrt{5}), \text{ taking } g=32.$$

This fall of 5 feet takes place in 20 minutes, $=20\times 60$ seconds.

Hence $\qquad 5 = \tfrac{1}{2}\cdot\frac{8A}{\pi}(\sqrt{10}+\sqrt{5})\times 20\times 60,$

from which $A = \dfrac{5\pi}{4\times 20\times 60\times(\sqrt{10}+\sqrt{5})}$ of a square foot.

By rationalizing the denominator, replacing π by 22/7, and multiplying the fraction by 144, we find that the effective area of the hole is $33\times(\sqrt{10}-\sqrt{5})/350, = \cdot 087$ of a square inch.

EXAMPLES XIX.

[In these Examples it is assumed that Torricelli's Theorem is exactly true.]

(*The answers are given on page* 338.)

1. A cylindrical vessel is placed with its axis vertical, and is kept full of water. If the height of the cylinder is 10 feet, and a small orifice is made 1 foot from the top of the cylinder, where will the jet strike the horizontal plane through the base of the cylinder?

2. If a small orifice is made in the side of the cylinder at a point midway

between the base and the top, show that the distance from the side of the cylinder of the point where the jet strikes the horizontal plane through the base of the cylinder is equal to the height of the cylinder.

3. A vessel contains water, and a small orifice is made in the side of the vessel at a depth of 4 feet below the surface, which is maintained at a constant level. If the effective area of the orifice is 1/2 of a square inch, find the number of cubic feet of water that will flow out of the vessel in an hour.

4. A cylinder, 16 feet high, is filled with water, and placed with its axis vertical. A small hole, whose effective area is 1/1000 of the area of the section of the cylinder is made in the base of the vessel, and the cylinder is kept full of water. In how many minutes will there be discharged from the cylinder a quantity of water equal to that which the cylinder holds?

5. A vessel, in the form of a cone, whose axis is 9 feet, is placed with its axis vertical and filled with water. A small hole is opened at the lowest point of the vessel, and when the vessel is kept full of water the quantity of water which the vessel can hold is discharged from the hole in 10 minutes. Compare the area of the hole with the area of the surface of the water.

6. A cylindrical vessel is 16 feet high, and its base, which is horizontal, has a diameter of 3 feet. It is filled with water, and a small hole, 1/4 of a square inch in area, is made in the bottom. What is the rate of descent of the surface (i) immediately after the hole is made; (ii) when the depth of the water is 9 feet; (iii) when the depth of the water is 4 feet?

(iv) After how many minutes will the cylinder be half-emptied; and (v) after how many minutes will the water all be discharged?

7. If the space within a closed vessel is a vacuum, and a small hole is made in the side of the vessel, the air will begin to rush into the vacuum with the velocity $\sqrt{2gH}$, where H is the height of the homogeneous atmosphere and g is the acceleration of gravity.

Give a numerical result, being given that the pressure of air is 14·88 lbwt. per square inch, and that a cubic inch of air weighs ·31 of a grain.

CHAPTER XVII.

SECTION I.—UNIFORM CIRCULAR MOTION.

169. Angular Velocity.

Def. *When a point is moving in any manner in a plane, the rate at which the straight line joining the moving point to a fixed point is rotating is called the angular velocity of the moving point about the fixed point.*

The angular velocity may be *uniform* or *variable.* It is uniform if the straight line joining the moving point to the fixed point turns through equal angles in equal times, and is variable if this line does not turn through equal angles in equal times. For example, the angular velocity about the centre of a circle of a point which describes the circumference with uniform speed is uniform.

The *measure of a uniform angular velocity* is the number of units of angle through which the rotating line turns in unit of time. In measuring angular velocity the unit of angle is usually taken to be the *radian,** and the unit of time to be the *second,* so that angular velocity is usually expressed in *radians per second.* We shall always suppose that angular velocity is expressed in these units.

The *measure of a variable angular velocity* at a given instant of time is the angle in radians through which the rotating line would turn in one second, if during that time the angular velocity remained constant and equal to its value at the instant under consideration.

In the case in which a rigid body rotates about a fixed axis, *e.g.* a door rotating about the line joining the hinges, all points of the body move in circles whose planes are perpendicular to the axis, and whose centres lie on the axis. And it is evident that the lines joining the several points of the body to the centres of the circles in which the points respectively moving, all turn through the same angle in the same time. Hence, at a given instant, the angular velocity is the same for all points of the body. This angular velocity is called the *angular velocity of the body about the fixed axis.*

170. Kinematics of Uniform Circular Motion.

If a point is describing the circumference of a circle with uniform speed, it will pass over equal arcs in equal times, and therefore the angular velocity of the point about the centre will be uniform.

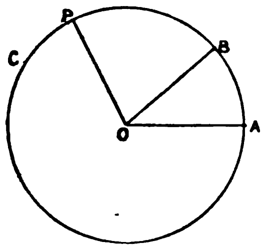
Let ABC be a circle, radius r feet; and let V be the velocity

* See the author's *Trigonometry,* chap. i.

in feet per second of a point describing the circumference uniformly. Let ω denote the angular velocity, and T the time in seconds of a revolution; then

$$\omega = V/r = 2\pi/T.$$

Let AB be the arc described by the revolving point in one second, so that AB is V feet; then the angle AOB is the angle which the line joining the moving point to the centre turns through in one second. Hence the number of radians in the angle AOB is the measure of the angular velocity, that is, is equal to ω.



But by trigonometry

arc AB = number of radians in angle AOB × radius,

or $V = \omega r$,

from which $\omega = V/r$,

which proves the first formula for ω.

Again, since T is the time of one revolution, we get, by the formula for uniform motion,

$$VT = \text{length of circumference,}$$
$$= 2\pi r \qquad \text{(Art. 14.)}$$

Hence $\omega r \cdot T = 2\pi r$,

from which $\omega = 2\pi/T$,

which proves the second formula for ω.

Ex. 1.—A point is moving in the circumference of a circle of r feet radius with a uniform angular velocity of ω radians per second. What time does it take to go over s feet of its path?

Since the angular velocity is ω and the radius is r, the velocity of the point in the circumference of the circle is ωr feet per second, and therefore the time taken by the point to move over s feet of its path is $s/\omega r$ seconds.

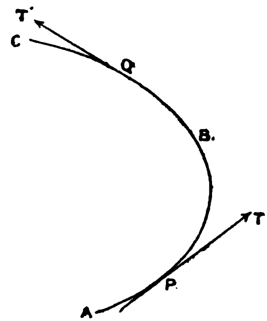
Ex. 2.—P is a point of a body turning uniformly round a fixed axis, and PN is a line drawn from P at right angles to the axis; if PN describes an angle of 375° in 3 seconds, what is the angular velocity of the body? and if PN is 6 feet long, what is the linear velocity of P?

In one second the body turns through an angle of 375/3, = 125 degrees or $25\pi/36$ radians. Hence the angular velocity of the body is $25\pi/36$, and the

linear velocity of P is $(25\pi/36) \times 6, = 5\pi/6$ feet per second. Taking $22/7$ as the value of π, we find the linear velocity of P to be $13\frac{2}{21}$ feet per second.

171. Centrifugal Force.

When a body is moving in a curve, the direction of motion is continually changing. Thus, if the curve is ABC, the direction of motion at any point P is the direction of PT, the tangent to the curve at P; and, similarly, the direction of motion at the point Q is the direction of QT′, the tangent at Q. Thus, while the body moves from P to Q, the direction of motion changes from the direction of PT to that of QT′.



Now, the first law of motion states that a body maintains its state of uniform motion in a straight line until it is acted on by force; and the second law of motion states that when a change takes place in the motion of a body, the force which produces the change must act in the direction in which the change takes place. Hence it follows that in order to cause a body to move in a curve, there must act on the body at every point of the curve a force which shall overcome the tendency of the body to maintain its motion in a straight line, and shall urge the body away from the tangent to the curve. This force may be applied either as a tension or as a pressure, but it must act at every point so as to drive the body from the tangent towards the concave side of the curve. For example, when a stone is attached to the end of a string and whirled round in a horizontal circle, the pull of the string on the stone is the force which is continually deflecting the stone from the tangent, and is causing the stone to move in the circle. Similarly when a train is moving round a curve, the reaction of the outer rails on the wheels of the carriages is the force which is continually changing the direction of motion of the train.

It was at one time supposed that when a body was moving in a circle there was a force acting from the centre outwards on

the body, tending to drive the body away from the centre, to which the name of **Centrifugal Force** was applied. And it was supposed that the force required to cause the body to move in the circle was the force which balanced the centrifugal force.

We now know that centrifugal force is not a force, but the property of matter now called *inertia*, in virtue of which a body tends to maintain its motion in a straight line. The term centrifugal force is, in fact, doubly misleading. For, in the first place, inertia is not a force; and, secondly, the body in virtue of its inertia tends to move, not perpendicular to the tangent, but along the tangent.

Although the term *centrifugal force* is based on a false conception, it is still in common use in dynamics as a convenient term to denote the tendency of a body which is moving in a circle to maintain its motion along the tangent. And it is still usual to use the language of old writers on dynamics, and to speak of the force which causes the body to move in a circle as the force balancing the centrifugal force. Thus, for example, in the case of the stone whirled round in a horizontal circle, we may say that the tension of the string balances the centrifugal force of the stone, or that the tension of the string is due to this centrifugal force. So also in the case of the train we may say that the pressure of the carriages on the outer rail is caused by the centrifugal force of the train moving in the curve. In the same way we may say that it is on account of centrifugal force that the balls in Watt's governor for the steam-engine recede from the rod to which they are hinged, when the rod is rotating rapidly.

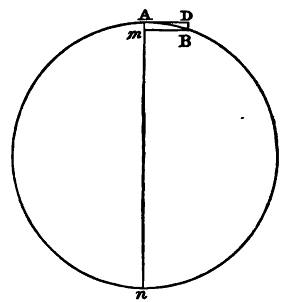
172. Centrifugal Force of a Body Moving Uniformly in a Circle.

When a body is moving in a circle with uniform speed, the force which is causing the body to move in the circle must at every point of the circle act along the radius, and towards the centre, and must be constant in amount. For since there is no change in the magnitude of the velocity, the force which acts on the body cannot have a component along the tangent, and must, therefore, act along the radius. Also, since the rate

of change of the direction of motion is uniform, the force which causes this change must be a uniform force.

Let the body be describing the circle A$D$$n$, radius r, with the uniform speed V. Let m be the mass of the body, and F the force which causes the body to move in the circle. Then, by the second law of motion, if the force F acted on the body *in a constant direction* it would produce an acceleration of the velocity of the body in that direction equal to F/m. Let this acceleration be denoted by a.

Let AB be an infinitesimal arc described by the body in the infinitesimal time t. Let An be the diameter of the circle drawn through A, and Bm the perpendicular on An. While the body is moving over the arc AB it is acted on by the constant force F, whose direction is continually changing. Since, however, the time t is very small, we may consider that the direction also is constant and parallel to An. Thus we may consider the motion during the time t to be the same as that of a body projected from A along the tangent AD with the velocity V, and moving from A to B with an acceleration a in the direction Am. Also since AB is very small, we may take the length of the arc AB to be equal to Bm.



Hence $\quad Bm = \text{arc AB} = Vt$, by the formula for uniform motion,

and $\quad Am = \frac{1}{2} at^2$, by the formula for uniformly accelerated motion.

But $\quad Bm^2 = Am \cdot mn$ (Euclid III. 35),

and therefore $V^2t^2 = \frac{1}{2} at^2 \cdot mn$;

from which $\quad a = 2\, V^2/mn.$

Now the smaller is Am, the more nearly do the suppositions we have made agree with the actual conditions of the motion. But the smaller is Am, the more nearly is mn equal to the

diameter An. Hence, if in the last-written equation we write $2r$ for mn, we shall arrive at the *exact* formula for a, viz.,

$$a = V^2/r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Hence, when a body is moving with uniform speed V in a circle of radius r, the body has at every instant an acceleration V^2/r towards the centre of the circle.

We can express this acceleration in terms of the angular velocity about the centre, ω say, and also in terms of the time of one revolution, T say.

Since $V = \omega r$, we have

$$\begin{aligned} a &= V^2/r = \omega^2 r^2/r, \\ &= \omega^2 r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \end{aligned}$$

Also since $\omega = 2\pi/T$, we can write

$$a = (2\pi/T)^2 \,.\, r \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The force F which causes this acceleration is equal to ma. Hence we have the following three equivalent expressions for F—

$$\begin{aligned} F &= mV^2/r \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)' \\ &= m\omega^2 r \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)' \\ &= m(2\pi/T)^2 . r. \ldots\ldots\ldots\ldots\ldots (3)' \end{aligned}$$

These expressions for F give the value in absolute units of the force which is required to cause the body to move in the circle. It is convenient to use the language referred to in the preceding article, and to say that they are expressions for the force which balances the centrifugal force, or to say that they are *expressions for the centrifugal force.* In fact, no error will be introduced by treating the body describing the circle as if it were in statical equilibrium under the action of the force causing the motion—a force mV^2/r towards the centre—and the fictitious centrifugal force of equal amount acting in the opposite direction.

The expressions (1)′, (2)′, (3)′ will give the value of the centrifugal force in *poundals* if m is expressed in pounds, r in feet, T in seconds, V in feet per second, and ω in radians per second.

Similarly if C.G.S. units are used, the expressions will give the value of the centrifugal force in *dynes*.

Ex. 1.—A toy car, whose mass is 1/2 of a lb., runs at the rate of 5 miles an hour on a level circular railway of 3 feet radius. Calculate the horizontal pressure on the rails.

The horizontal pressure of the car on the rails is equal and opposite to the horizontal component of the reaction of the rails on the car. But it is the latter force that causes the car to move in the circle.

Hence the horizontal pressure of the car on the rails

$$=mV^2/r, \ =\tfrac{1}{2}\times(\tfrac{22}{3})^2\times 3, \ =8{\cdot}963 \text{ poundals.}$$

Taking g to be 32, this is a force of about ·28 of a pound weight.

Ex. 2.—A stone, whose mass is 200 grammes, is swung round in a horizontal circle 3 times in a second at the end of a string 109 centimetres long. What is the tension of the string in grammes weight?

[Take $g=981$ cm./sec.^{2} and $\pi^2=9{\cdot}87$.]

The tension of the string balances the centrifugal force, and is therefore equal to $m\,.\,4\pi^2r/T^2g$ grammes weight, where

$$m=200,\ r=109,\ T=1/3,\ g=981,\ \pi^2=9{\cdot}87.$$

Substituting these numbers, we find that the tension of the string is equal to the weight of 7896 grammes.

EXAMPLES XX.

[Take $\pi=22/7$, $\pi^2=10$, and $g=32$ ft./sec.^{2}]

(*The Answers are given on page* 338.)

1. A point is moving uniformly in a circle with an angular velocity of 3 radians per second. How many revolutions is the point making per minute?

If the radius of the circle is 10 feet, what is the linear velocity of the point?

2. A body moves in a circle with a uniform linear velocity of 20 ft./sec., and passes over a quadrant of the circle in 3 seconds, what is the angular velocity about the centre in radians per second, and what is the radius of the circle?

3. A body describes a circle, the length of whose circumference is c, in time t, with uniform angular velocity ω about the centre. Show that the linear velocity is c/t, and that the radius of the circle is $c/t\omega$.

4. Two points are describing circles uniformly with linear velocities inversely proportional to the square roots of the radii of the circles. Show that the squares of the times of revolution are proportional to the cubes of the radii.

5. A stone is swung round in a horizontal circle, first at the end of one

string and then at the end of another twice as long, so as to make the same number of revolutions in a minute in both cases. Compare the tensions of the two strings.

6. What force is required to keep a body whose mass is 10 lbs. rotating in a circle of 15 feet radius, (i) with a velocity of 30 ft./sec., (ii) if it is making one complete revolution in a second?

7. The breaking stress of a wire is 100 lbwt. If a mass of 10 lbs. is fastened to a piece of this wire, and swung round in a horizontal circle of 20 feet radius, what velocity will the 10 lbs. have when the wire breaks?

8. A mass of 2 lbs. is whirled round on a smooth horizontal plane by a thread 4 feet long attached to a point in the plane. If the greatest mass which the thread can support without breaking is 9 lbs., what is the greatest velocity which can be given to the mass so as not to break the thread?

9. What is the force in grammes weight required to keep a mass of 20 grammes revolving 30 times in a second in a circle of 4 centimetres diameter?

[Take $g = 981$ cm./sec.^{2}]

10. A particle whose mass is 10 lbs. is constrained to move in a horizontal circle by a string 5 feet long fastened to a fixed point. If at any instant the tension of the string is 98 poundals, find the velocity of the particle, and its angular velocity about the fixed point.

11. If a rigid body is rotating about a fixed axis the energy of rotation is $Mk^2\omega^2/2$, where ω represents the angular velocity of the body, and Mk^2 the moment of inertia of the body about the axis.

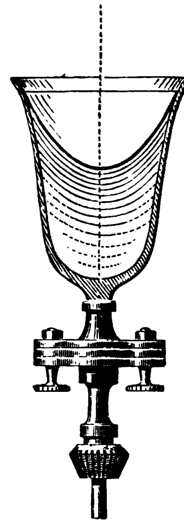
12. A fly-wheel weighs 10000 lbs., and is of such a size that the matter comprising it may be treated as if concentrated on the circumference of a circle 12 feet in radius. What is its kinetic energy when moving at the rate of 15 revolutions a minute? How many turns would it make before coming to rest if the steam were cut off, and it moved against a friction of 400 lbs. exerted on the circumference of an axle of 1 foot in diameter?

Section II.—ROTATING LIQUID.

173. Form of the Surface of a Liquid in a Rotating Vessel.

When an open vessel containing liquid is made to rotate, the surface of the liquid will not remain horizontal. We shall inquire what the form of the surface is in the case in which the vessel is rotating uniformly about a vertical axis. The liquid and the vessel will ultimately rotate about the axis as if they formed parts of a solid body. Each particle of the liquid will describe a horizontal circle, whose centre lies in the axis, and whose radius is equal to the length of the perpendicular from

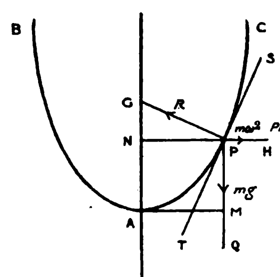
any position of the particle upon the axis of rotation; and the angular velocity of each particle about the centre of the circle in which it is moving will be the same as that of the rotating vessel.



It follows that the surface of the liquid will be in the form of a surface of revolution, and that all planes through the axis will cut the surface of the liquid in equal and similar curves. We shall show that the form of any section of the surface, made by a plane through the axis, is a *parabola* whose vertex is downwards, and whose axis coincides with the axis of rotation. The form of the section is shown in the accompanying figure, the dotted line representing the axis of rotation.

It will follow that the surface of the liquid will be the surface generated by the rotation of a parabola about its axis. A surface of this form is called a *paraboloid of revolution.*

Let BAC be a section of the surface of the liquid, made by a plane drawn through the axis of rotation ANG, A being the point where the axis meets the surface of the liquid. It is evident that the curve BAC must be symmetrical with respect to the line ANG. Let P be any point of the surface of the liquid in the plane of the section, and let PN be the perpendicular from P upon the axis, so that NPH is horizontal. Let PQ be the vertical through P, SPT the tangent to the curve BAC at the point P, and PG the normal at the same point, that is, the line perpendicular to the tangent. Let ω denote the



angular velocity of the rotating liquid, and g the acceleration of gravity.

Consider the forces acting on a particle of liquid, of mass m, at the point P. The particle is moving with uniform angular velocity ω in a horizontal circle, whose radius is PN, under the action of two forces—(i) Its weight mg acting vertically downwards, that is, in the direction PQ; and (ii) the resultant pressure of the surrounding liquid on the particle, a force which we shall denote by R. The resultant of these two forces is the force causing the particle to move in the circle, and this resultant must therefore (by formula (2)′, Art. 172) be equal to $m\omega^2 \, . \, \mathrm{PN}$, and must act from P to N. Or we may say that the resultant of the weight, mg, and the pressure of the liquid on the particle, R, is in equilibrium with the fictitious centrifugal force $m\omega^2 \, . \, \mathrm{PN}$, acting from N to P.

Hence we may consider the particle to be in statical equilibrium under the action of its weight mg, acting from P to Q, the centrifugal force $m\omega^2 \, . \, \mathrm{PN}$, acting from P to H, and the resultant pressure of the surrounding liquid R. And it follows from the fundamental property of a fluid that R must act perpendicular to the surface at P. For the surface in the neighbourhood of the point P is an element of an inclined plane of liquid (of which SPT is the section), on which the particle of liquid at P may be taken to be at rest under the action of its weight, the centrifugal force, and the reaction of the inclined plane. But this reaction must be perpendicular to the inclined plane, since there is no tangential force between two parts of a fluid which are in relative equilibrium (Art. 53). Hence the force R must act perpendicular to SPT, that is, along the normal PG.

We thus arrive at the result that a particle of liquid at P is in equilibrium under the action of three forces, mg, $m\omega^2 \, . \, \mathrm{PN}$, and R, acting from P to Q, from P to H, and from P to G respectively. Hence, by the triangle of forces, these forces must be proportional to the lengths of the sides of the triangle GNP, whose sides are in the directions of PQ, PH, and PG respectively.

Therefore

$$NG : NP = mg : m\omega^2 . PN$$

from which $$NG = g/\omega^2.$$

Hence the form of the section must be such that whatever be the position of the point P, the length of NG is constant. NG is the projection of the normal PG on the axis, and is called the *subnormal*, so that the curve BAC is such that the subnormal is constant. It is known from geometry that the only curve in which the subnormal is constant is the *parabola*. Hence the section of the surface by any plane through the axis of rotation is a parabola of which ANG is the axis, and the point A the vertex.

Since the latus rectum of a parabola is double of the subnormal, the latus rectum of the curve BAC is $2g/\omega^2$, and the equation connecting the lengths AN and PN is, by conic sections,

$$PN^2 = (2g/\omega^2) . AN.$$

174. Pressure at any point of the Liquid.

Since the particles of the liquid are in relative equilibrium, it follows that the stress between any two parts of the liquid must be nórmal to the surface separating the two parts. Hence we may assume, as in the case of a liquid at rest (Art. 56), that the pressure at a given point of the rotating liquid is the same in all directions. But on account of the stress produced in the liquid by the centrifugal forces, the pressure will not be the same at all points of the same horizontal plane, but will be greater the greater the distance from the axis of rotation.

Let Q (see figure, page 285) be any point in the rotating liquid, and let QMP be the vertical line through Q, meeting the surface of the liquid in the point P. Let w denote the weight in gravitational units of unit volume of the liquid, and let p denote the intensity of pressure at the point Q, measured in gravitational units of force per unit of area. Then a column of liquid in the form of a cylinder of very small section whose axis is PQ, and whose base is horizontal, may be considered to be in equilibrium under the action of the pressure upward on the

base, the weight of the liquid, the pressure of the liquid on the curved sides, and the centrifugal force of the mass of liquid in the column. Of these forces the pressure on the base and the weight of the liquid act vertically, and the other forces act horizontally. Hence we get, as in Prop. II. Art. 61,

$$p = w \,.\, \mathrm{PQ}.$$

Hence the intensity of pressure at a point of a mass of rotating liquid is equal to the intensity of hydrostatic pressure due to a depth equal to the distance, measured vertically, of the point below the surface of the liquid.

When g, the acceleration of gravity, and ω, the angular velocity of the rotating liquid are given, the equation to BAC is known. Hence when AH or PN, the horizontal distance of Q from the axis is given, the length of AN, the height of P above A, can be calculated. If therefore the position of Q with reference to A is known, PQ, the depth of Q below the surface, may be calculated, and the intensity of pressure at Q may be found.

If the foot, the second, and the weight of a pound are taken as the units of space, time, and force respectively, then ω will be expressed in radians per second, PQ in feet, g in feet and seconds, and w in pounds weight per cubic foot. The formula for p will then give the intensity of pressure in pounds weight per square foot.

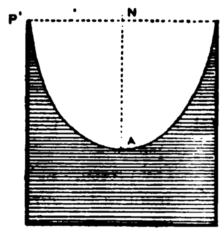
Ex. 1.—A cylindrical open vessel, 1 foot high, is placed with its axis vertical and is *filled* with water. The vessel is then made to rotate about its axis at the rate of 2 revolutions per second. If the diameter of the section of the vessel is 1 foot, find (i) the depth of the lowest point of the surface of the water below the top of the cylinder; (ii) the quantity of water that will be spilt; (iii) the pressure in lbwt. per square foot at a point of the side of the cylinder 2 inches below the lowest point of the surface.

[Take $g = 32$, and $\pi^2 = 10$.]

When the vessel is set rotating, some water will be spilt, and the surface will become concave. A section through the axis will be a parabola PAP′, with AN, the axis of the cylinder, as its axis, and with the point A as its vertex. The curve will pass through P and P′, the points on the rim of the vessel which lie in the plane of the section.

(i) To find AN, the depth of the lowest point of the surface below PP′.

Since P is a point on the curve, from the equation to the curve, viz., $PN^2 = (2g/\omega^2) \,.\, AN$, we can solve for the length AN in terms of the known quantities PN, ω, and g. Thus—



$$AN = \omega^2 PN^2/2g,$$

where $g = 32$, $PN = \frac{1}{2}$, and $\omega = 4\pi$. (Art. 170.)

Hence $AN = (4\pi)^2 \times \frac{1}{4}/64, \; = \pi^2/16, \; = 10/16,$
$= 5/8$ of a foot, $= 7\frac{1}{2}$ inches.

(ii) To find the quantity of liquid that is spilt.

The volume of water that is spilt is the volume of the concavity PAP′, which is a segment of a paraboloid of revolution. It is a known result that the *volume of a segment of a paraboloid of revolution is half the volume of a cylinder on the same base and of the same height.* Hence the quantity of water that is spilt is the quantity that would fill a length of $3\frac{3}{4}$ inches of the cylinder. The ratio of this quantity to the quantity that fills the cylinder is $3\frac{3}{4} : 12$, or $5 : 16$. Hence 5/16 of the original quantity of water in the cylinder will be spilt.

(iii) To find the pressure at a point of the side of the cylinder 2 inches below A.

Since the point is 2 inches below A, it is $9\frac{1}{2}$ inches below P, and the pressure at the point is the pressure due to a height of $9\frac{1}{2}$ inches of water. Taking a cubic inch of water to weigh 250 grains, the pressure of $9\frac{1}{2}$ inches of water is $9\frac{1}{2} \times 250 = 2375$ grains per square inch, that is, about 1/3 of a pound weight per square inch.

Ex. 2.—In the figure of page 285 show that if θ is the slope of the surface at the point P,

$$\tan \theta = \omega^2 \,.\, PN/g.$$

The slope of the surface at any point is the angle which the tangent plane to the surface at that point makes with the horizon. In the figure the slope of the surface at P is the angle HPS, which is equal to the angle NGP. Hence

$$\tan \theta = PN/NG, \; = PN/(g/\omega^2),$$
$$= \omega^2 PN/g.$$

As a numerical example let us calculate the slope of the surface at the highest point of the surface of the liquid in the case considered in Ex. 1.

Here $\omega = 4\pi$, and $PN = 1/2$.

Hence $\tan \theta = 16\pi^2 \times \frac{1}{2}/32, \; = \pi^2/4,$
$= 2{\cdot}5$, (taking $\pi^2 = 10$).

From a table of natural tangents we find that the angle whose tangent is 2·5 is about 68°. Hence the slope of the surface at the point P is an angle of 68°

EXAMPLES XXI.

[Take $g=32$ ft./sec.^{2}, $\pi=22/7$, and $\pi^2=10$.]

(*The Answers are given on page* 338.)

1. A cylinder, whose radius is one foot, is *partly filled* with water, and turns uniformly 3 times a second about its axis, which is vertical. Find—

(i) The depth of the lowest point of the surface of the water below the highest point.

(ii) The pressure at a point of the side of the cylinder 6 inches below the lowest point of the surface.

(iii) The pressure at a point 4 inches from the side of the cylinder, and 6 inches above the lowest point of the surface.

(iv) The pressure at a point 8 inches from the side of the cylinder and 4 inches below the lowest point of the surface.

(v) The slope of the surface at a point 6 inches above the lowest point, being given tan $73° = 3{\cdot}33$.

2. A cylinder, placed with its axis vertical, is *filled* with water, and is made to rotate about its axis. If the radius of the cylinder is 1 foot 9 inches, and the height $7\frac{9}{16}$ feet, how many revolutions per second must the vessel make in order that half the quantity of water in the vessel may be spilt?

3. A cylinder, whose radius is 6 inches, contains water to a depth of 10 inches. The cylinder is made to rotate about its axis, which is vertical, at the rate of 2 revolutions per second. What is the height of the highest point of the surface of the water above the base of the cylinder, assuming that the cylinder is so high that no water is spilt?

What is the least height of the cylinder for which this condition will be fulfilled?

4. A cylinder, whose axis is vertical, is *partly* filled with water. Show that when the cylinder is made to rotate about its axis and no water is spilt, there will be no change in the *total* pressure on the base, but that there will be an increase in the total pressure on the sides. Is the pressure on the base uniform when the vessel is rotating?

5. Calculate the *increase* in the total pressure in lbwt. on the sides of the cylinder in Ex. 3.

6. An open cylindrical vessel, whose axis is vertical, is *filled* with water, and is then made to rotate about its axis. Show that there will be no change in the total pressure on the sides of the cylinder, and that there will be a diminution in the total pressure on the base, equal to the weight of water that is spilt.

CHAPTER XVIII.

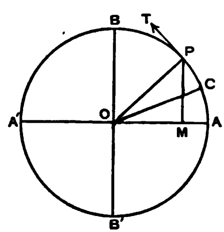
Section I.—HARMONIC MOTION.

175. Simple Harmonic Motion.

Def. *If a point moves in a straight line so that the acceleration of its velocity is directed towards a fixed point in the line, and proportional to its distance from that point, the motion of the point is a simple harmonic motion.*

Prop. If a point P is describing the circumference of a circle with uniform velocity, and if PM be drawn perpendicular to AA′, a fixed diameter of the circle, then the motion of the point M is a simple harmonic motion.

We shall suppose that the motion of P is in the direction opposite to that of the hands of a watch. It is evident that the motion of M will be an oscillatory motion. When P is at A, M is at A; and as P moves over the first quadrant AB of the circle, measured from A, M will move along the radius AO, and will be at O when P is at B. While P is describing the second quadrant BA′, M will move from O to A′, and P and M will again coincide at A′. Thus while P is describing the first half ABA′ of the circumference of the circle, M is moving from A to A′. Similarly, while P is describing the second half of the circumference, A′B′A, M is moving from A′ back to A. Thus the motion of M is an oscillatory motion between the extreme positions A and A′.



Let a denote OA, the radius of the circle, T the time of revolution of P, V the linear velocity of P, and ω its angular velocity about the centre. Let the figure represent the position of P at any time, and let OM, the distance of M at that time from O, be denoted by x.

We shall find expressions in terms of these letters for the velocity and acceleration of M.

The *velocity* of M is the same as the component of the velocity

of P in the direction AA′, and is equal to the product of the velocity of P and the cosine of the angle which the direction of motion of P makes with AA′.

Hence the velocity of M

$$= V \times \text{cosine of the angle which PT makes with OA},$$
$$= V \,.\, \sin \mathrm{AOP} = V \,.\, \mathrm{PM}/a,$$
$$= (V/a)\sqrt{a^2 - x^2} \ldots\ldots\ldots\ldots \quad (1).$$
$$= \omega \sqrt{a^2 - x^2} \ldots\ldots\ldots\ldots \quad (2), \text{ (since } V = a\omega\text{)}.$$

The *acceleration* of M is the component in the direction MO of the acceleration of P. But the acceleration of P is V^2/a in the direction PO (Art. 172).

Hence the acceleration of M is always directed towards O, and is

$$= (V^2/a) \,.\, \cos \mathrm{AOP} = (V^2/a) \,.\, \mathrm{OM}/a,$$
$$= V^2 x/a^2 \ldots\ldots\ldots\ldots \quad (3),$$
$$= \omega^2 x^2 \ldots\ldots\ldots\ldots \quad (4), \text{ (since } V = a\omega\text{)}.$$
$$= 4\pi^2 x/T^2 \ldots\ldots\ldots\ldots \quad (5), \text{ (since } \omega = 2\pi/T, \text{ Art. 170)}.$$

Hence the acceleration of M is always directed towards O, and is proportional to the distance of M from O. The motion of M is therefore a *simple harmonic motion.*

Formula (1) shows that when $x = a$, that is, at A and A′, the velocity of M is zero; that when $x = o$, that is, at O, the velocity is greatest and equal to V; and that the velocity is less the greater the value of x, that is, the greater the distance of M from O. Again, formulæ (3) and (4) show that at A and A′ the acceleration is greatest, and equal to V^2/a, $= \omega^2 a$; and that the acceleration at O is zero.

The general character of the motion of M during a revolution of the point P in the circle will now be apparent. In the first quarter of the revolution, as P moves from A to B, M moves from A to O with a velocity which increases from zero to V, and with an acceleration which decreases from V^2/a to zero. In the second quarter of the revolution, as P moves from B to A′, M moves from O to A′ with a velocity which falls from V to zero, and with an acceleration, opposite to that of the direction of motion, which increases from zero to V^2/a. In the third quarter of the revolution, as P moves from A′ to B′, M moves back from A′ to O with a motion which is equal and opposite to the motion of M in the second quarter; and in the last quarter, as P moves from B′ to A, M moves from O to A with a motion which is equal and opposite to the motion of M in the first quarter.

Cor. 1. Let a point M be executing a simple harmonic motion in a straight line with an acceleration equal to $n^2 \times$ distance of moving point from a fixed point in the line, and let a be the greatest distance of the moving point from the fixed point. Let a circle be described with the fixed point as centre, and with a as radius. Then the motion of M is the same as the motion of the projection on the straight line of a point P which moves in the circumference of this circle with uniform angular velocity about the centre equal to n.

For by the above proposition the projection of P upon the straight line moves in that line with a simple harmonic motion, the acceleration at any distance x from the centre of the circle being, by formula (4), equal to n^2x. Hence the motion of the projection of P is the same as the motion of M.

Cor. 2. If a mass is executing simple harmonic vibrations, the *force causing the motion* will be found for a given position by multiplying the mass by the acceleration in that position. Hence the force must vary as the distance from the mean position.

176. Displacement, Amplitude, Period, Epoch, and Phase of a Simple Harmonic Motion.

Let a point M be executing a simple harmonic motion in the straight line AA′, and let O be the middle point of AA′. (See figure, page 291.) Then O is called the *mean position* of the moving point. Let the acceleration of M in any position be $n^2 \times$ distance from the mean position; then the motion of M is the same as the motion of the projection of a point P describing the circle on AA′ as diameter with uniform angular velocity n.

Certain terms, defined as follows, are used in the consideration of the motion of M :—

Def. The *displacement* of M in any position is the distance of M from the mean position, that is, the distance OM.

It is convenient to distinguish between the two directions in which the point M may be displaced from O by using the terms *positive* and *negative* with the corresponding signs + and −.

We shall suppose that the direction from O to A is chosen as the direction of positive displacement.

Def. The *amplitude of a simple harmonic motion* is the greatest displacement on either side of the mean position, that is, the distance OA or OA′.

Def. The *period of a simple harmonic motion* is the interval of time in which the moving point passes from one extremity of its range to the other, and back again to the first extremity.

This time, which is also called the *periodic time*, or the *time of a complete oscillation*, or the *time of a complete vibration*, is equal to the time of revolution of P in the circle ABA′B′.

The position of the moving point M will be known when the corresponding position of the point P is known. The position of P at a given instant may be specified in terms of (i) the time that has elapsed between a certain fixed point of time and the given instant, and (ii) the position of P at the fixed point of time. Calling this fixed point of time the *zero of time*, let C be the position of the moving point P at this zero, and let t denote the time that elapses while the point moves from C to P (see figure). Then the position of the moving point M will be known when the time t and the angle AOC (or the time of describing it) are known. The angle AOC is called the *epoch in angle*, and the time of describing it the *epoch in time*. The time during which the moving point M passed from A to M is called the *phase* of the motion at the time t. Hence the following definitions:—

Def. The *epoch in time* of a simple harmonic motion is the interval of time during which the moving point passes from its position of greatest displacement in the positive direction to its position at the zero of time. The *epoch in angle*, measured in *radians*, is a fraction of 2π whose numerator is the epoch in time and whose denominator is the periodic time.

Def. The *phase* of a simple harmonic motion at any given instant is the time during which the moving point moved from its position of greatest displacement in the positive direction to its position at the given instant, this time being expressed as a fraction of the periodic time.

177. Formulæ for the Periodic Time and for the Displacement.

Let T be the periodic time, β the epoch in angle, measured in radians, a the amplitude, and x the displacement at time t of a point describing a simple harmonic motion with acceleration equal to $n^2 \times$ displacement.

Then $$T = 2\pi/n \text{..............................(1).}$$
and $$x = a \cos(nt + \beta) \text{......................(2).}$$

For, by Cor. 1, Art. 175, the periodic time is equal to the time of revolution of a point describing the circumference of a circle with uniform angular velocity about the centre equal to n, and is therefore (Art. 170) equal to $2\pi/n$. Hence *the periodic time is independent of the amplitude.*

Taking the figure of page 291, the displacement when the moving point is at M is OM. Therefore

$$\begin{aligned} x &= \text{OM}, = \text{OP} \cos \text{MOP}, \\ &= \text{OA} \cos (\text{COP} + \text{AOC}), \\ &= a \cos (nt + \beta), \end{aligned}$$

the angle AOC being the epoch in angle, β, and the angle COP being the angle described by the revolving radius OP in time t with angular velocity n, or nt radians.

If the epoch is zero, that is, if the time t is measured from the instant wher the moving point is at A, then

$$\begin{aligned} x &= a \cos nt, \\ &= a \cos \frac{2\pi t}{T} \text{..........................(3).} \end{aligned}$$

by formula (1) above.

The discussion of S. H. motion derives its importance from the application of the results to many questions in physics. We shall see in Section II. of this chapter that the motion of particles of air in a tube along which an aerial pulse is travelling consists of simple harmonic vibrations of small amplitude through their mean positions.

EXAMPLES.

1. A point is executing simple harmonic vibrations with an amplitude of 1/100 of a foot, the number of complete vibrations in one second being 200. Find—

(i) the velocity at the mean position;

(ii) the velocity at a point midway between the centre and an extreme point;

(iii) the acceleration at an extreme point;

(iv) the acceleration at a point midway between the centre and an extreme point.

Here $a=1/100$, $T=1/200$, $n=2\pi/T$, $=400\pi$.

Hence the motion is the same as the motion of the projection on a diameter of a point describing a circle, radius 1/100 of a foot, with uniform angular velocity 400π radians per second about the centre.

(i) The velocity at the mean position, by formula (2), Art. 175,

$$=an, =(1/100) \,.\, 400\pi,$$
$$=4\pi \text{ ft./sec.}$$

(ii) The velocity at a point midway between the centre and an extreme position, by same formula,

$$=n\sqrt{a^2-x^2}, =400\pi\sqrt{(1/100)^2-(1/200)^2},$$
$$=2\pi\sqrt{3} \text{ ft./sec.}$$

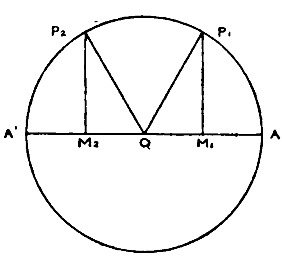
(iii) The acceleration at an extreme point, by formula (5), Art. 175,

$$=4\pi^2 a/T^2, =4\pi^2 \,.\, (1/100)\div(1/200)^2,$$
$$=1600\pi^2 \text{ ft./sec.}^2$$

(iv) The acceleration at a point midway between the centre and an extreme point, is half the result in (iii), since the displacement is halved, that is, is $800\pi^2$ ft./sec.^{2}

2. The amplitude of a simple harmonic vibration is 6 feet, and the period is 4 seconds. How long does the moving point take to go from a point 3 feet on one side to a point 3 feet on the other side of the mean position, and what are the velocity and acceleration at either of these points?

Draw a line AA′ to represent a length of 6 feet, and take O its middle point. Let M_1 and M_2 be the middle points of OA and OA′ respectively; and on AA′ describe a circle. Take AA′ to represent the line in which the point is executing vibrations, so that the motion of the point in AA′ is the same as the motion of the projection on AA′ of a point describing the circle uniformly in the same time.



The time taken by the point moving in AA′ to go from M_1 to M_2 is equal to the time taken by the point moving in the circle to go from the point P_1 to P_2. But since OM_1 is 3 feet, and OP_1, $=OA$, is 6 feet, therefore the angle M_1OP_1 is an angle of 60°. Similarly the angle M_2OP_2 is 60°; and therefore the angle P_1OP_2 is 60°. Since this

angle is one-sixth of 4 right angles, the time taken by the point moving in the circle to move from P_1 to P_2 is one-sixth of the time of revolution, that is, is 2/3 of a second. This, therefore, is the time taken by the point moving in AA′ to go from M_1 to M_2.

The angular velocity of the point moving in the circle is $2\pi/4, = \pi/2$ radians per second. Hence the velocity at M_1 or M_2 is, by formula (2), Art. 175,

$$= (\pi/2)\ \sqrt{6^2 - 3^2},\ = (\pi/2)\sqrt{27},$$
$$= 3\pi\sqrt{3}/2 \text{ ft./sec.}$$

Also the acceleration at either of these points is, bv formula (5), Art. 175,

$$= 4\pi^2 \, . \, 3/16,$$
$$= 3\pi^2/4 \text{ ft./sec.}^2$$

3. A point is executing S.H. vibrations with an amplitude of 1/10 of a foot, and a period of 1/50 of a second. Find (i) the velocity at the mean position; (ii) the velocity at a point 1/30 of a foot from the mean position; (iii) the acceleration at an extreme point; (iv) the acceleration at a point 1/40 of a foot from the mean position.

Ans. (i) 10π ft./sec. (ii) $20\pi\sqrt{2}/3$ ft./sec. (iii) $1000\pi^2$ ft./sec.^{2} (iv) $250\pi^2$ ft./sec.^{2}

4. A particle, whose mass is 1/2 of an ounce, is performing S.H. vibrations with an amplitude of 1/100 of a foot, and a period of 1/256 of a second. Find in lbwt. (i) the greatest force on the particle; (ii) the force at a distance of 1/300 of a foot from the mean position. [Take $\pi^2 = 10$.]

Ans. (i) 25·6. (ii) 8·53.

5. A point is performing S.H. vibrations in a period of T seconds. If V denote the velocity (in feet per second) of the point at its mean position, show that at a *time* t seconds after passing through the mean position—

(i) the displacement is $(VT/2\pi) \, . \, \cos\,(2\pi t/T)$;

(ii) the velocity is $V \, . \, \sin\,(2\pi t/T)$;

(iii) the acceleration is $(2\pi V/T) \, . \, \cos\,(2\pi t/T)$.

SECTION II.—AERIAL WAVES.

178. Propagation of an Aerial Wave in a Straight Tube.

Let a piston be fitted into one of the ends of a long straight tube which contains air, and let it be kept moving rapidly forwards and backwards in the tube through a very small distance. We shall suppose that each complete vibration is performed in the same time, and that the amplitude of each vibration is the same.

This motion of the piston will cause the air which is immediately in front of it to be alternately compressed and rarefied. While this air is being compressed, it compresses the air in front of it, and the latter in like manner produces the same effect on the next portion of air; and so on. Thus, while the piston is moving forwards a disturbance passes through the air in a certain length of the tube, in consequence of which this air is in a state of condensation. While the piston is moving backwards the disturbance produced in the forward motion of the piston continues to pass through the air in the tube, and in the same time the air which at the end of the forward motion of the piston was in a state of compression now becomes rarefied. Thus, at the end of a complete vibration of the piston forwards and backwards in the tube, the air in a certain length of the tube in front of the piston is in a state of rarefaction, and the air in an equal length of the tube in front of this rarefied air is in a state of compression. The lengths of these rarefied and compressed portions of air will be very great compared with the distance through which the piston is vibrating.

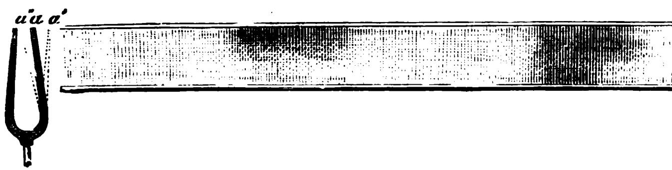
The condensation and rarefaction produced during any vibration of the piston will continue during the subsequent vibrations to travel with uniform velocity along the tube, and will be followed by the condensations and rarefactions produced by these vibrations. Thus the following effects will be produced by the vibrations of the piston:—

(i) The air *in a given part of the tube* will in successive equal intervals of time be alternately compressed and rarefied.

(ii) The air in the whole length of the tube will *at a given instant of time* be divided up into portions of equal lengths which are alternately in a state of condensation and in a state of rarefaction.

The same effects would be produced in the air in a tube by simply holding a vibrating tuning-fork with one of its prongs close to an open end of the tube. This is illustrated in the accompanying figure, in which a is the mean position of the vibrating prong, a' the position in which it is nearest to the tube, and a'' the position in which it is farthest from the tube.

While the prong is moving forwards from a'' to a' the air in the mouth of the tube is compressed, and while the prong is moving backwards from a' to a'' this air is rarefied, the conden-



sations and rarefactions so produced following each other along the tube.

The figure represents the state of the air in successive portions of the tube at the instant when the vibrating prong is in its mean position and approaching the tube. In those parts of the tube in which the shading is darker the air is in a state of condensation, and in those in which the shading is lighter the air is in a state of rarefaction, the density of air being greatest at the middle point of a condensed portion of the air and least at the middle point of a rarefied portion.

The condensation and rarefaction in two adjacent portions of the air in the tube form an *aerial wave.* Each complete vibration of the tuning-fork will give rise to an aerial wave, which will be propagated along the tube with a velocity of about 1100 feet per second.

When aerial waves enter the ear, they produce vibrations in the drum of the ear, and a *sound* is heard. The figure may therefore be taken to be a representation of the state of air (with respect to density) in a tube along which waves of sound are travelling.

179. Motion of the Particles of Air constituting an Aerial Wave.

Let us imagine that there is placed at any point in the tube a very light thin disc, which is capable of moving freely to and fro in the tube; and let us consider how this disc will move when a wave of condensation and rarefaction passes it.

While the condensation is passing the disc the pressure of

the air behind the disc is greater than the pressure of the air in front of it, and in consequence the disc moves forwards in the tube through a very small distance. On the other hand, while the rarefaction is passing the disc the pressure of the air behind the disc is less than the pressure of the air in front of it, and in consequence the disc moves back again in the tube. Hence during the passage of the wave of condensation and rarefaction the disc performs a complete vibration of small amplitude forwards and backwards in the tube.

Now consider at any point of the tube a very thin layer of air, bounded by two planes perpendicular to the length of the tube and very close together. This layer of air will move in the tube during the passage of an aerial wave in nearly the same way as if it were a solid disc. Hence while a wave of condensation and rarefaction is passing a given point of the tube, the particles in a thin layer of air at that point make a complete vibration of small amplitude forwards and backwards in the tube. It is proved in the mathematical theory that the motion of each particle of air is a simple harmonic motion about its mean position.

The student must carefully distinguish between *the motion of the wave* and *the motion of the particles of air constituting the wave.* A particle of air oscillates to and fro in the tube through a very small distance on either side of its mean position, the amplitude depending on the amplitude of the vibrating tuning-fork. This may be only a very small fraction of an inch. The motion of the particles of air at a given part of the tube produce condensations and rarefactions, which are propagated along the tube, the propagation of these states or conditions of the air (with respect to density) forming the essential part of the motion of an aerial wave.

The velocity with which a wave travels along the tube is found both by theory and from experiment to be independent of the amplitude of vibration of the particles of air. It depends on the ratio of the pressure of air in the tube to the density, and is therefore constant for a constant temperature. It is about 1120 feet per second at the temperature of 60° F., and increases with the temperature. This is therefore the velocity with which waves of sound travel through the air at the temperature of 60° F.

The *intensity* or loudness of a sound depends on the amplitude of the vibrating particles of air in the sound waves.

When the vibrating body, *e.g.* the vibrating tuning-fork, from which sound waves proceed, is making the same number of vibrations per second,

the same number of sound waves would enter the ear in each second, and a *musical note* would be heard. The *pitch* of the note depends on the number of vibrations per second which the sounding body is performing.

In this and the preceding article we have described the characteristics of aerial waves, which differ essentially from water waves, waves of light, &c.

180. Wave Length of an Aerial Wave.

The sum of the lengths of the condensed portion and the rarefied portion of an aerial wave is called the *length of the wave.* It is evidently equal to the distance from the centre of one condensation (or rarefaction) to the centre of the next condensation (or rarefaction).

It follows from the preceding articles that the wave length is the distance through which the front of the wave travels during the time of one complete vibration of a particle of the air.

Let v denote the velocity of propagation of the wave, T the time of vibration of a particle of air, and λ the length of the wave. Then, since the front of the wave travels with uniform velocity v, it follows that in time T it travels over the distance vT.

Hence $$\lambda = vT.$$

The velocity v is constant as long as the temperature is constant. The time T is equal to the time of a complete vibration of the body, a tuning-fork suppose, whose motion gives rise to the wave.

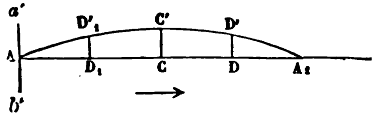
Thus. for example, if a tuning-fork is making 256 vibrations per second, then T is 1/256 of a second. Taking v to be 1120 feet per second, the length of the waves of sound produced by the vibrations of the tuning-fork will be 1120/256 feet, that is, about $4\frac{1}{2}$ feet.

181. Geometrical Representation of the Motion of the Particles of Air in an Aerial Wave.

The motion of the particles of air in an aerial wave may be represented graphically in the following way.

Take AA_1 to represent a length of the tube in which the air at a given instant of time is in a state of condensation. Then AA_1 is half a wave length. The condensation will be greatest at C, the middle point of AA_1, and for other points in AA_1 will be less the greater the distance from C. Now the

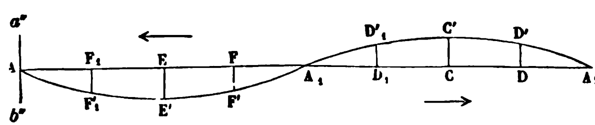
degree of condensation at any point depends on the displacement of the vibrating particles of air at that point from their mean positions, the condensation being greatest where the displacement of the particles is greatest and least where the displacement is least. Thus at A and A_1 the particles of air will be in their mean positions, at C the particles will be at their positions of greatest displacement in advance of their mean positions, and at D and D_1 the particles will be in a position intermediate between their mean positions and their positions of maximum displacement. If we now suppose that a perpendicular is erected at each point in AA_1, and that the lengths of the perpendiculars are made proportional to the displacements of the vibrating particles from their mean positions, the extremities of these perpendiculars will be points of a curve $AD'_1C'D'A_1$. This curve will therefore represent graphically the displacements of the vibrating particles of air at all points in a condensed part of an aerial wave.



The student must bear in mind that while the wave length of an aerial wave may be a certain number of feet, the maximum displacement of a vibrating particle of air is in general a very small fraction of an inch. Thus CC' will in general be very small compared with AA_1.

In the same way we may draw a curve which shall represent graphically the displacements of the vibrating particles of air in the rarefied part of a wave.

The accompanying figure is a graphical representation of the displacements of the particles of air in a complete wave. AA_1



represents the part of a wave in which the air is in a state of rarefaction, and A_1A_2 the part in which the air is in a state of condensation. As indicated by the arrow-heads, the particles

of air in AA_1 are displaced backwards from their mean positions, while those in A_1A_2 are displaced forwards from their mean positions. The displacements of the particles of air in AA_1 are represented by perpendiculars drawn downwards from AA_1, and the displacements of the particles in A_1A_2 are represented by perpendiculars drawn upwards from A_1A_2. The ends of the perpendiculars form the curve $AE'A_1C'A_2$, which represents graphically the displacements of the particles at the different points of a complete aerial wave.

Let a denote the amplitude of the vibrating particles of air, and T the time of a complete vibration. Let z denote the displacement of any particle at the t, this time being measured from the instant when the particle last passed through its mean position in the forward direction. Then, assuming that the motion of the particle is a simple harmonic motion, it follows from Art. 177 that

$$z = a \sin (2\pi t/T).$$

Since $$\lambda = vT \text{ (Art. 179)},$$

the formula for z may be written in the form

$$z = a \sin (2\pi vt/\lambda).$$

For a given value of t, the particle will be in advance of or behind its mean position according as the value of z given by this formula is positive or negative.

182. Interference of Aerial Waves.

Let two aerial waves of equal wave lengths be travelling respectively in two tubes A and B, which open into a third tube C.

If the waves from A and B enter C in the same phase, so that the condensation of one coincides exactly with the condensation of the other, and the rarefaction of one with the rarefaction of the other, they will combine to form in C a wave, in which the amplitude of the vibrating particles of air will be the sum of the amplitudes of the vibrating particles in the component waves.

But if the waves from A and B enter C in opposite phases, so that the condensation of one coincides exactly with the rarefaction of the other, they will form in C a wave, in which the amplitude of the vibrating particles will be the difference

of the amplitudes of vibration in the component waves. And if the phases differ by less than half the time of a complete vibration of a particle of air in either of the waves, the two component waves will form in C a wave, in which the amplitude of the vibrating particles will be less than the sum, and greater than the difference, of the amplitudes of the vibrating particles in the component waves. These are simple cases of the phenomenon of *interference* of two aerial waves.

The phenomenon of interference of aerial waves is illustrated in the following experiment:—A vibrating tuning-fork is held close to one end A of a tube, divided near that end into two branches which join each other again near the other end B of the tube. Thus the waves of sound which proceed from the tuning-fork enter the tube and travel along the two branches. If the lengths of the two branches are the same, the waves which travel along the two branches will be in the same phase at the junction B of the two branches. But if the lengths differ by half a wave length, the waves from the two branches will meet each other in opposite phases. Now it is found that when the two branches are of the same length, the sound which issues from the end B of the tube is of much greater intensity than the sound that is heard when the vibrations of the tuning-fork pass directly from the tuning-fork to the ear. On the other hand, the more and more nearly do the lengths of the two branches of the tube differ by half a wave length, the more and more feeble is the sound that is heard to issue from the end B of the tube. This shows clearly that the more nearly two waves which meet each other are in opposite phases, the more nearly do the waves destroy each other.

The result of the interference of two aerial waves may be deduced from the formula of Art. 181 for the displacement of a particle of air in an aerial wave.

For when two waves meet each other, the displacement of any particle of air from its mean position will be the sum of the separate displacements due to the component waves.

Taking the case in which the waves proceed from the same vibrating body, let a denote the amplitude of vibration, and T the time of vibration of the vibrating particles of air in either of the component waves. Also

let e denote the difference of phase of the two waves when they meet each other. Then if t denotes the time measured from the instant when the disturbance due to the wave which is in advance of the other first reached a given particle, the resultant displacement of that particle at time t will be

$$=a \sin(2\pi t/T)+a \sin\{2\pi(t-e)/T\} \text{ (Art. 181)}$$
$$=2a \sin\{2\pi(t-\tfrac{1}{2}e)/T\}.\cos(\pi e/T) \text{ (by trigonometry).}$$

If $e=0$, that is, if the waves are in the same phase, $\cos(\pi e/T)=1$, and the displacement is $2a \sin(2\pi t/T)$. This shows that the resultant displacement will be double the displacement due to one of the component waves. Hence in the resultant wave the amplitude of vibration is double the amplitude of vibration in one of the component waves.

If $e=T/2$, that is, if the waves are in opposite phases, $\cos \pi e/T, =\cos \pi/2, =0$, and the resultant displacement is zero for all values of t. Hence the two waves destroy each other.

CHAPTER XIX.

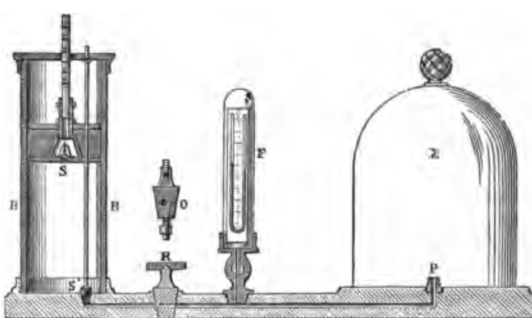
HYDROSTATIC AND PNEUMATIC APPLIANCES.

Air-Pumps, Arts. 183 to 186.

183. The Single-barrelled Air-pump.

The air-pump is used for withdrawing the air from a closed vessel.

The accompanying figure represents a single-barrelled pump.



The vessel E, called the *receiver,* from which the air is to be withdrawn, is placed mouth downwards on a metal plate in the centre of which is a small aperture P. From this aperture

a passage, opening by a *valve* at S′, leads into the barrel BB of the pump. In the barrel a *piston*, fitted at S with a *valve* which opens upwards, works air-tight. The barrel is not closed at the top, so that the space above the piston always contains air at atmospheric pressure.

Suppose the piston is being raised from the bottom of the cylinder. The valve S in the piston will be closed by the outside atmospheric pressure, and at the same time the valve S′ will be opened by the pressure of the air in the passage PS′. Thus on the upward stroke of the piston the pressure of air in the receiver and the passage PS′ will force some air into the barrel of the pump.

On the downward stroke of the piston the volume of air under the piston will be reduced, so that the pressure of this air will increase, and will soon exceed the pressure of air in the receiver. The valve S′ will then close, and the pressure of air in SS′ will increase until it exceeds the atmospheric pressure. When this stage has been arrived at the valve S in the piston will open upwards, and the air in SS′ will escape through this valve into the outer atmosphere.

Thus in an upward stroke of the piston the barrel will be filled with air from the receiver, and in the succeeding downward stroke this air will escape. The air in the receiver will thus become more and more rarefied, and there is theoretically no limit to the degree of exhaustion that may be arrived at. In practice it is found that, due to the imperfections of the machine, the rarefaction soon reaches a certain limit, which cannot be passed by continuing to work the pump.

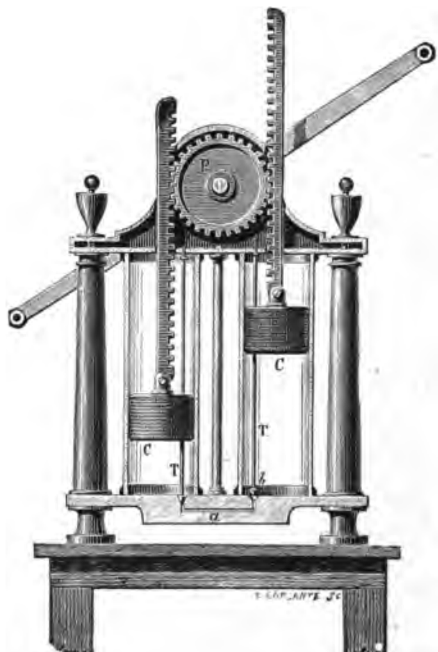
In some machines the valve S′ is in the form of a narrow slit covered by a flap of silk. When the pressure of the air on the upper surface of the flap exceeds the pressure on the lower surface, the silk is pressed closely on the slit, and thus prevents the passage of air. When the pressure on the under surface is the greater, the flap is raised and air passes freely through the slit. In the machine shown in the figure the valve S′ is a metal stopper, which by a mechanical arrangement is raised and lowered along with the piston.

In the figure R is a *stop-cock*, by turning which air can be readmitted into the receiver.

A *mercurial gauge* F serves to indicate the degree of exhaustion that has been attained at any time. The gauge differs from a siphon barometer only in the shorter length of the closed branch. The open branch is in communication with the receiver, and before the pump is worked the closed branch is full of mercury. When the pump is worked the pressure of air in the receiver is reduced; and when the pressure falls below that due to a column of mercury whose length is equal to the height of the closed branch above the surface of mercury in the open branch of the gauge, the mercury will fall in the closed branch and rise in the open branch. After this stage in the exhaustion has been passed, the difference of the heights of the mercury in the two branches is a measure of the pressure of air in the receiver.

184. The Double-barrelled Air-pump.

The single-barrelled air-pump has two defects. (i) During the downward stroke of the piston no air is being withdrawn from the receiver. (ii) During the upward stroke work is being done against the resultant downward pressure of air on the piston. This resultant is the *excess* of the downward pressure of the atmosphere on the upper surface of the piston above the upward pressure of the air in the barrel on the under surface. As the exhaustion proceeds the pressure of the air in the barrel diminishes, and therefore the pump becomes harder to work the greater the degree of rarefaction in the receiver.



The first defect is completely remedied in the double-barrelled

air-pump, in which there are two barrels fitted each with a piston, and both in communication with the receiver. The pistons are worked up and down by the same lever, one piston ascending while the other is descending. By this arrangement there is no idle stroke, and air is withdrawn continuously from the receiver.

The other defect of the single-barrelled pump is also remedied in the double-barrelled pump. For while the pressure of the atmosphere resists the upward motion of the ascending piston, it at the same time assists the downward motion of the descending piston. It follows therefore that, on the whole, no work is done against atmospheric pressure when the pump is worked.

It is easily seen, however, that at no part of the stroke in a double-barrelled pump will the pressure of air in the two barrels be the same. For the air under the descending piston is being compressed, and as the piston descends the pressure of this air increases until it becomes equal to the atmospheric pressure. At the same time the air under the ascending piston is becoming more rarefied, so that its pressure becomes less as the piston ascends. Hence the upward pressure resisting the motion of the descending piston in one barrel is always greater than the upward pressure assisting the motion of the ascending piston in the other barrel. A certain force is required, therefore, to work the pump.

185. **Rate of Exhaustion.**

It is easy to compare the density of air in the receiver after a given number of strokes of the piston with the original density.

Let V denote the volume of the barrel, and V' the volume of the receiver.

When the piston is raised, the air which at first occupied the receiver now occupies the receiver and barrel. If then ρ_0 denotes the original density of air in the receiver, and ρ_1 the density after one stroke, it follows (Art. 23) that

$$\rho_1 (V+V') = \rho_0 V';$$

from which

$$\rho_1 = \rho_0 V'/(V+V')$$
$$= k\rho_0, \text{ where } k = V'/(V+V').$$

Similarly, if ρ_2 denote the density at the end of the second stroke $\rho_2 = k\rho_1, = k^2\rho_0$.

In general, if ρ_n denote the density after the nth stroke

$$\rho_n = k^n\rho_0,$$
$$= \rho_0 V'^n / (V + V')^n.$$

Also, since by Boyle's law the pressure varies as the density, we get

$$p_n = p_0 V'^n / (V + V')^n,$$

where p_0 is the original pressure of air in the receiver, and p_n is the pressure after n strokes.

The student will notice that the word "stroke" in this article means a complete stroke in a single-barrelled pump, and *half* a complete stroke in a double-barrelled pump.

Ex. 1. If the pressure of air in a receiver is reduced to one-third of the atmospheric pressure in 4 strokes, to what would it be reduced in 6 strokes?

Here p_0, the original pressure, is the atmospheric pressure. Denote this by the number 1, and let k denote the fraction $V'/(V+V')$. Then since after 4 strokes the pressure is reduced to 1/3, we get $k^4 = 1/3$, and therefore $k^2 = 1/\sqrt{3}$.

Hence the pressure after 6 strokes

$$= k^6, = (k^2)^3, = (1/\sqrt{3})^3,$$
$$= \frac{1}{3\sqrt{3}}, = \frac{\sqrt{3}}{9}, = \frac{1{\cdot}732}{9} = \frac{19}{100} \text{ nearly.}$$

Hence the pressure after 6 strokes would be 19/100 of the atmospheric pressure.

Ex. 2.—The receiver of a single-barrelled air-pump has five times the volume which the barrel has, and the length of the stroke of the piston is 12 inches. Find how far the piston must descend after the third upward stroke before the valve in the piston will rise, the weight of the valve being neglected.

Here $V' = 5$, $V = 1$, $p_0 = 1$ atmosphere, and therefore p_3, the pressure of air in the receiver and barrel after the third upward stroke, is $(5/6)^3, = 125/216$ of the atmospheric pressure.

The valve will rise on the descent of the piston after the third stroke when the pressure of air under the piston becomes equal to the atmospheric pressure. Let x represent the distance through which the piston must fall before the valve rises. The volume of air in the barrel before the piston descends is measured by 12, and the volume when the value rises by $12 - x$.

The corresponding pressures are measured by 125/216 and 1. Hence by Boyle's law

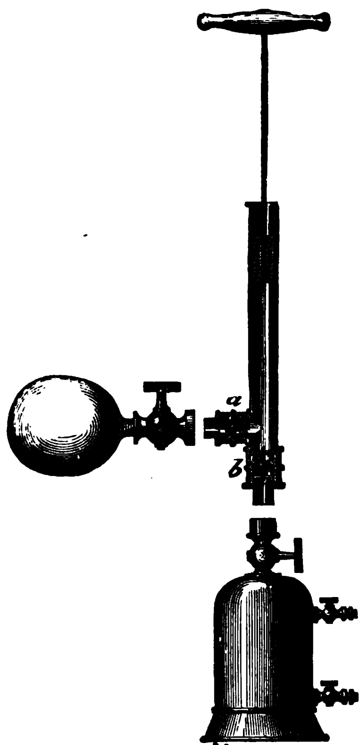
$$125/216 = (12 - x)/12.$$

Solving this equation, we find that x, the required distance, is $5\frac{1}{18}$ inches.

186. The Condenser.

The compression pump or condenser is a machine for forcing air or other gas into a closed vessel.

The condenser is an air-pump in which the valves are made to open inwards instead of outwards, so that at every stroke of the piston the air is forced into the receiver.



The accompanying figure shows the arrangement of parts in a condenser. At a point in the side of the cylinder in which the piston works there is a valve *a* opening inwards, and at the bottom of the cylinder there is a valve *b* which opens into the receiver. During the upward stroke of the piston the pressure of air in the receiver closes the valve *b*, and the atmospheric pressure opens the valve *a*, so that when the upward stroke is completed the cylinder contains air at atmospheric pressure. On the downward stroke the pressure of air in the cylinder closes the valve *a* and opens the valve *b*, so that the air in the cylinder is forced into the receiver. It is evident that in each complete stroke there is forced into the receiver the same quantity of air, viz. the quantity required to fill the cylinder at atmospheric pressure.

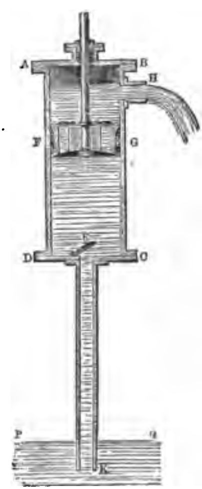
By putting the cylinder in communication at *a* with the gas in a closed vessel—the vessel shown on the left in the figure—the gas in the vessel will be forced by the condenser into the receiver.

Pumps for Water, Arts. 187 to 189.

187. The Common or Suction Pump.

In the common pump a piston FG, containing a valve opening upwards, works air-tight in a cylinder ABCD, at the lower part of which is an aperture fitted with a valve E, which opens upwards. From this aperture a pipe EK leads down to water in the well, the end K of the pipe dipping below the surface PQ of the water.

When the piston FG is drawn upwards the pressure of the atmosphere on the valve of the piston will close that valve and a vacuum will be left below the piston. The pressure of the air in the pipe EK will therefore open the valve E. Thus the air will flow from the pipe EK into the space below the piston, and the air which at first occupied the volume of the pipe EK, will, when the piston is at its highest point, occupy the volume of the pipe EK and of the space DCGF. The pressure of the air in the pipe EK will thus be reduced below the atmospheric pressure, and consequently the atmospheric pressure on the surface PQ of the well will force some water up the pipe EK. When the piston descends, the valve E will close, while the valve in the piston will open and allow the escape of the air in the space DCGF.



On continuing to work the pump the pressure of air in the pipe EK will become less and less, and the water in that pipe will rise higher and higher until it rises to E. So far the pump acts virtually as an air-pump. On still continuing to work the pump, the water will flow through the valve E on the upward stroke of the piston, will be forced upwards into

the space FGAB through the valve in the piston on the downward stroke, and will issue from an aperture H in the side of the cylinder.

Since it is the pressure of the atmosphere which forces the water up the pipe KE into the cylinder, it is evident that the depth of the surface PQ of the well below DC must not be greater than the height of the water barometer.

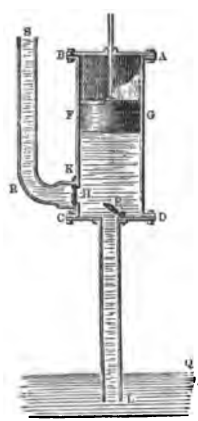
Force necessary to work the Pump.

Let A denote the area of the piston in square feet, h the height of the columm of water GH above it in feet, and h' the height of the column of water GK below it. Then the force necessary to work the pump

$$\begin{aligned}&= \text{force downwards on piston} - \text{force upwards}\\ &= \text{atmospheric pressure on piston} + Awh\\ &\quad - (\text{atmospheric pressure on piston} - Awh')\\ &= Aw(h+h'),\end{aligned}$$

w being the weight of a cubic foot of water.

This result shows that the force required to work the pump is equal to the weight of a column of water, whose sectional area is equal to the area of the piston, and whose height is equal to the height of the aperture H above the surface PQ of water in the well.



188. The Force Pump.

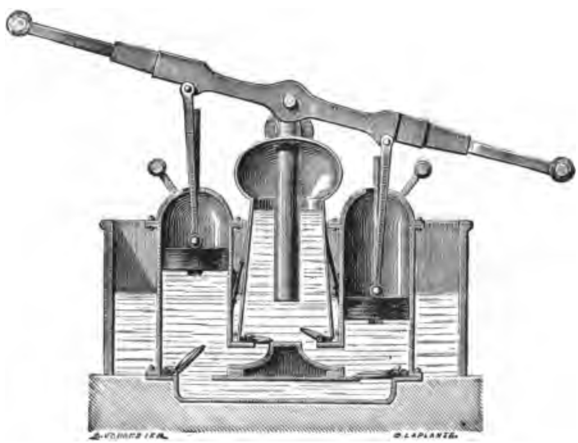
In this pump a solid piston FG works in a closed cylinder ABCD, between the limits B and K. The aperture H through which the water issues is below the point K, and is fitted with a valve which opens outwards. In other respects this pump does not differ from the common pump.

The explanation of the working of the pump is similar to that given of the working of the common pump. After the pump has been worked for some time the pressure of the atmosphere forces the water up the pipe LE to the valve E.

If the piston is now raised the water will rise into the cylinder DCGF, and on the downward stroke of the piston the valve E closes, the valve H opens, and the water in DCGF is forced up the pipe RS. The valve H serves the purpose of preventing the return of the water in the pipe RS to the cylinder of the pump when the piston is on the upward stroke.

189. The Fire-Engine.

In the fire-engine two forcing pumps force water into a common reservoir, the upper part of which, called the air-



chamber, contains compressed air. The pressure of this air acting on the surface of the water in the reservoir gives a steady flow of water through the outlet pipe which dips under the surface of the water. The pipe supplying the water to the machine is attached to a water main. This pipe is not shown in the figure.

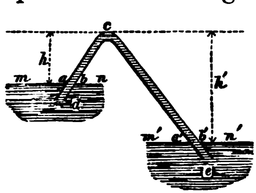
190. The Siphon.

The siphon consists of a bent tube open at both ends. It is used for withdrawing the liquid from a vessel, as shown in the accompanying figure.

In order to transfer liquid by means of the siphon, the tube

is first filled with the liquid, and the ends are then closed. One end of the tube is held below the surface of the liquid in the vessel, the other end being outside the vessel. If the

latter end of the tube is at a lower level than the surface of the liquid in the vessel, on opening the ends of the tube the liquid will flow through the siphon, and will issue from the end of the tube which is outside the vessel.



The siphon will continue to work when the end of the tube outside the vessel is under the surface of liquid in another vessel, as in the accompanying figure, as long as this surface is below the level of the surface of the liquid in the first vessel. Thus if mn and $m'n'$ are the surfaces of the liquid in the two vessels respectively, the liquid will continue to flow through

the siphon *dce*, in the direction from *d* to *e*, as long as the level of *mn* is higher than the level of *m'n'*.

The action of the siphon depends on the pressure of the atmosphere. Let us take a section of the tube at the highest point *c*, and let us consider the forces acting at *c* in the directions *ce* and *cd* when the ends *d* and *e* are opened. The pressure in the direction *ce* is the pressure of the atmosphere on the surface *mn* of the liquid, communicated through the liquid to *c*, *less* the weight of a column of the liquid whose height is equal to the height, *h*, of *c* above the surface *mn*. Also the pressure at *c* in the direction *cd* is the pressure of the atmosphere on the liquid at the end *e*, communicated through the liquid to *c*, *less* the weight of a column of the liquid whose height is equal to the height, *h'*, of *c* above the surface *m'n'*.

If H denote the height of a barometer filled with the liquid, the pressure in the direction *ce* is due to a height $H - h$ of the liquid, and in the direction *ca* that due to a height $H - h'$ of the liquid. Hence the pressure in the direction *ce* exceeds the pressure in the direction *cd* by that of a height $H - h - (H - h')$ or $h' - h$ of the liquid. Since *h'* is greater than *h*, the pressure in the direction *ce* is greater than the pressure in the direction *cd*. Hence the liquid at *c* will flow down the part *ce*, and the atmospheric pressure on *mn* will force liquid up the part *dc* to supply its place. A continuous stream will thus flow from *d* to *e*.

The siphon will continue to work as long as *h'* is greater than *h*, that is, as long as the level *m'n'* is lower than the level *mn*.

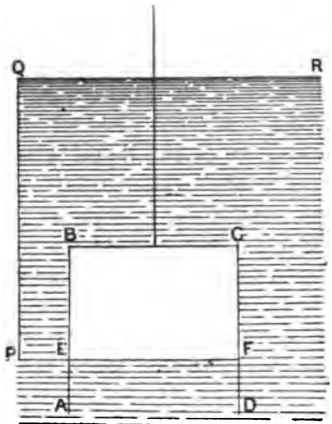
Also it is evident that the siphon will not work at all if *h* is greater than H, that is, if the height of *c* above *mn* is greater than the height of the barometer filled with the liquid.

191. The Diving-bell.

The diving-bell is a large vessel ABCD constructed of iron, closed at the top BC and at the sides AB and CD, but open at the bottom AD.

When the bell is let down into water, the air in the bell will tend to exclude the water. At any depth below the surface

the air in the bell is subject to the pressure of the water, and is therefore compressed. As the bell descends, the pressure of the water increases, and therefore by Boyle's law the volume of air in the bell becomes less.



Let the bell be at such a depth that the level EF of the water in the bell is at a depth PQ below the surface. Then the air in the bell is under the pressure of the depth PQ of water *plus* the atmospheric pressure on the surface of the water. If h denote the height of the water barometer, the air in the bell may be considered to be under the pressure of a depth $(PQ+h)$ of water. Now before the bell was sunk in the water the air was under the pressure of a depth h of water. Hence by Boyle's law

$$\frac{\text{volume of air in bell at depth PQ}}{\text{volume of the bell}}=\frac{h}{PQ+h}.$$

Ex.—A diving-bell, having a capacity of 125 cubic feet, is sunk in salt water to a depth of 100 feet. If the specific gravity of salt water is 1·02, and the height of the water barometer is 34 feet, find the total quantity of air at atmospheric pressure required to fill the bell.

At the depth of 100 feet in salt water the pressure is the sum of the atmospheric pressure and the pressure due to the depth of water. Hence the pressure is that due to a depth of $(34+1{\cdot}02\times 100), =136$ feet of *pure* water. Hence the quantity of air that fills the diving-bell is that quantity of air which occupies a volume of 125 cubic feet at a pressure of 136 feet of pure water.

It follows from Boyle's law that the same quantity of air would occupy at atmospheric pressure, that is, the pressure due to 34 feet of water, the volume

$$136\times 125/34, =500 \text{ cubic feet.}$$

Hence the quantity of air which must be forced into the diving-bell in order to completely fill the bell would occupy at atmospheric pressure a volume of 500 cubic feet.

EXAMPLES XXII.

(*The Answers are given on page* 338.)

1. The pressure of air in the receiver of an air-pump is a pressure of 30 inches of mercury. After a few strokes it is 27 inches. What part of the original contents of the receiver has been withdrawn?

2. If the pressure of air in the receiver of an air-pump is reduced to 1/16 of the atmospheric pressure in 4 strokes, to what pressure would it be reduced in 6 strokes?

3. A barometer is placed under the receiver of an air-pump, the mercury standing at 30 inches before the pump is worked. If 1 stroke of the pump brings it down to 20 inches, where will it stand after 3 strokes?

4. The volume of the barrel of an air-pump is 1/8 of that of the receiver, and the pressure of air in the latter is a pressure of 765·45 millimetres of mercury. To what will this be reduced after 3 strokes of the pump?

5. If the barometer stands at 29·7 inches, what will the mercurial gauge of an air-pump read when the quantity of air withdrawn is ten times as much as the quantity left in the receiver?

6. If each of the barrels of a double-barrelled air-pump has a volume of one-tenth of that of the receiver, what diminution of pressure will be produced in the receiver after 4 strokes of the handle of the pump?

7. In the common pump one foot of the length of the barrel of the pump holds a gallon of water, which may be taken to weigh 10 lbs. At each stroke the piston works through 8 inches, and the spout is 24 feet above the surface of the water in the well. How many foot-pounds of work are done in each stroke?

8. By placing at the spout of a common pump a valve opening into a pipe, the pump is converted into a lift pump. If a lift pump is employed to raise water to a height of 200 feet, what force (in addition to its weight) will be required to lift the piston, the area of the piston being 100 square inches? [A cubic foot of water weighs $62\frac{1}{2}$ lbs.]

9. If the piston or plunger in the forcing pump has a cross section of 4 square inches, and works 80 feet below a cistern, with what pressure must it be pushed down to force water into the cistern?

10. A vessel containing water is being emptied by means of a siphon. What effect would be produced by making a small hole through the top of the siphon?

11. Why is the flow in the common siphon more rapid—other circumstances being the same—the greater the difference in the lengths of the legs?

12. A cylindrical diving-bell, $7\frac{1}{2}$ feet high, is let down into water until its top is $7\frac{1}{2}$ feet below the surface of the water. To what height will the water rise in the bell, the water barometer standing at 34 feet?

13. A diving-bell, with a capacity of 200 cubic feet, rests on the bottom in water of 150 feet depth. If the height of the barometer is 29·5 inches, and the specific gravity of mercury 13·6, find how many cubic feet of air, at atmospheric pressure, are required to fill the bell.

14. A cylindrical diving-bell weighs 2 tons, and has an internal capacity of 200 cubic feet, while the volume of the material composing it is 20 cubic feet. The bell is made to sink by weights attached to it. At what depth may the weights be removed and the bell just not ascend—it being given that the mass of a cubic foot of water is 1000 ozs. and the height of the water barometer 33 feet?

MISCELLANEOUS EXAMPLES.

(*The Answers are given on page* 338.)

1. Two vertical cylindrical vessels of the same size and communicating with each other at the bottom are filled with water to a height of two feet. Half the water in one is then removed and replaced by oil of specific gravity ·7. Show how high the water will stand in each vessel when equilibrium has been restored.

2. A vessel with vertical sides has a horizontal base in the form of a square whose side is 1 inch. Find to what height the vessel must be filled with liquid in order that the resultant pressure on each vertical face may be equal to that on the base.

3. A quantity of heavy liquid is at rest under the action of gravity, its surface extending over a considerable area. Show what you know regarding (i) the form of the free surface; (ii) the relation between the pressures in different directions at any point; (iii) the magnitude of the pressure at different depths below the surface.

How might the value of "g" in different latitudes be determined by means of a delicate pressure-gauge immersed to a constant depth, say 20 feet, beneath the surface of still water?

4. Two cylindrical vessels of the same diameter and with vertical sides contain quantities of water and mercury respectively which produces the same pressure on the base of each. If s denotes the specific gravity of mercury, compare the total pressures on the sides.

5. What is the force tending to push out a cork of sectional area 6 square centimetres in the bottom of a vessel which contains layers of mercury, water, oil, of the respective depths 3, $3\frac{1}{2}$, 4 centimetres, the densities being respectively 13·6, 1, ·8 grammes per cubic centimetre?

6. A cube whose side is 1 foot is held with its upper face horizontal at a depth of 1 foot below the surface of sulphuric acid, the height of the barometer being 30 inches. Determine in lbwt. the pressures on the faces of the cube, taking atmospheric pressure into account.

[A cubic foot of water weighs 1000 oz., and the specific gravities of mercury and sulphuric acid are 13·6 and 1·2 respectively.]

7. Calculate the whole pressure on a cube just immersed in water, with two faces horizontal, its volume being 1000 cubic centimetres, taking account of atmospheric pressure which is that due to a depth of 1000 centimetres of water.

8. Down one arm of a U-tube containing water is poured some olive oil,

and the free surfaces of the water and the oil are noticed to be 69 and 75 centimetres respectively above the common surface. Turpentine is then poured into the other arm until its free surface is at the same level as the olive oil, and the depth of the turpentine is found to be 50 centimetres. Determine the densities with respect to water of olive oil and turpentine.

9. A cubical box is filled with water, and is fitted with a lid ABCD consisting of a uniform square plate, whose weight is 2/3 of that of the contained water. If the box is held so that the lid is inclined at an angle of 45° to the vertical, and the hinge AB is horizontal and above CD, show that the lid is on the point of opening.

10. A sphere whose internal diameter is 1 foot contains mercury which covers 3/4 of the vertical diameter. Find the stress across the horizontal plane passing through the centre of the square, assuming that a cubic inch of mercury weighs 3400 grains.

11. A right cylindrical vessel of weight W, the area of each of whose ends is A, has one of its ends open and the other closed, except for a tube of small bore a. Being placed with its open end on a smooth horizontal plane, water is poured in slowly through the tube in the top until the cylinder is full. A heavy piston, exactly filling the tube and capable of sliding freely through it is now gently inserted, and presses on the water. Find, in terms of W, A, a, the greatest weight of the piston that the water may not force up the cylinder so as to escape below it.

12. ABC is a triangle completely immersed in a liquid with the vertex C in the surface. Show how to divide the triangle into n parts by straight lines drawn through A, so that the pressure on each of the parts may be the same.

13. A square is immersed in a liquid with a diagonal vertical, and its highest point half as deep only as its lowest point. Find the centre of pressure.

14. A triangular area has one of its angular points on the surface of water, and its two other angular points at depths h and k respectively. Find the depth of the centre of pressure.

15. Show that the depth below the surface of a liquid of the centre of pressure of a rectangle, two of whose sides are horizontal, and at depths a and b respectively, is $2(a^2+ab+b^2)/3(a+b)$.

16. A rectangle, height h, is immersed vertically in water with one side in the surface. If a triangle whose base is the lower horizontal side, and vertex the middle point of the side in the surface of the liquid, is cut out of the rectangle, find the centre of pressure of the remainder.

17. On a vertical wall of a reservoir a circle, radius a, is described with its centre in the surface of the water. Find the centre of pressure of the part under water. [The distance of the centre of gravity of a semicircle, radius a, from the centre is $4a/3\pi$.]

18. A bucket in the form of the part of a cone contained between two

planes perpendicular to the axis is 1 foot high, the radii of the top and bottom being 6 and 4 inches respectively. The bucket is placed with its base horizontal, and is then filled with water, the weight of a cubic foot of which is 1000 oz. Find (i) the whole pressure, (ii) the resultant pressure on its curved surface.

19. A solid hemisphere whose radius is 3 inches is held under mercury with its base vertical and its centre 6 inches below the surface of the mercury. Assuming that the mass of a cubic inch of mercury is half a pound, determine in lbwt. (i) the pressure on the base, (ii) the resultant pressure on curved surface.

20. A hollow cone, whose vertical angle is a right angle, is held with its axis vertical and vertex downwards, and is filled with water. Find, in terms of h, the length of the axis, and w, the weight of unit volume of water, the resultant pressure of the water on either of the portions into which the curved surface is divided by a plane through the axis.

21. A right circular cone, height h, radius of base r, is filled with water, the weight of unit volume of which is w. It is then closed, and laid on a horizontal table with a generating line in contact with the table. Find (i) the resultant pressure on the base, (ii) the resultant vertical and horizontal pressures on the curved surface.

22. A composition is made of two metals A and B, the specific gravities of which are s_1 and s_2. The composition weighs a ounces in air and b ounces in water. Show that the ratio of the volume of A to the volume of B is $\{s_2(a-b)-a\} \div \{a-s_1(a-b)\}$.

23. The specific gravities of gold and silver are 19·4 and 10·5 respectively. An alloy of these metals weighs 979 ounces in vacuo, and 890 in water. Find the quantities of gold and silver in the alloy.

24. The mass of a body A is thrice that of a body B, but their apparent weights in water are the same. Given that the specific gravity of A is $5/3$, determine the specific gravity of B.

25. A cylinder, radius 5 centimetres, height 7 centimetres, and specific gravity 1·8, floats with its axis vertical in a liquid of specific gravity 2·18 contained in a cylindrical vessel of radius 9 centimetres. Find the height through which it must be raised in order to clear the liquid, and show that the work done in the process is 617400 π ergs.

[Take $g=981$ cm./sec.^{2}]

26. A solid displaces $1/a$, $1/b$, $1/c$ of its volume respectively when it floats in three different liquids. Find the volume which it displaces when it floats in a mixture (i) of equal volumes of the liquids, (ii) of equal weights of the liquids.

27. A body, weighing more than a kilogramme, is fastened to a spring-balance which is graduated up to a kilogramme weight only. When the body is immersed successively in a given liquid, in water, and in mercury, the tensions of the spring are 972, 984, and 228 grammes weight respectively. What is the specific gravity of the liquid, that of mercury being 13·6?

28. Given the densities of two substances A and B relatively to a third substance C, show how to calculate the density of A relative to B.

It is found that when 9 grammes of wax are placed in a cage of platinum wire, of mass 5·25 grammes, the weight of the two in water is 4 grammes. Find the specific gravity of wax, that of platinum being 21.

29. A uniform glass tube, whose length is 100 millimetres and internal and external radii 4 and 5 millimetres respectively, is closed at one end and floats vertically in water with 10 millimetres of its other end above the surface. Find the specific gravity of the glass.

30. A, B, C are three balls of equal weight. A balances B and C when all are suspended in a liquid of specific gravity s_1; B balances C and A in a liquid of specific gravity s_2; C balances A and B in a liquid of specific gravity s_3. Find the specific gravities of A, B, and C.

31. A cylindrical lead pencil floats in water with 7/8 of its volume immersed. If the lead is a cylinder whose radius is 1/4 of that of the pencil, and the specific gravity of wood is ·78, find the specific gravity of the lead.

32. A vessel contains olive oil standing on the top of water, and a cylinder of wood, so loaded as to float upright, floats partly in the water and partly in the oil, no part of it being unimmersed. The length of the loaded cylinder is $4\frac{1}{2}$ inches; its specific gravity is ·94; and the specific gravity of the oil is ·82. What length of the cylinder will be immersed in the water?

33. A vessel contains layers of equal thickness h of three different liquids, which do not mix, and whose densities are in arithmetical progression. A cone, the length of whose axis is $3\,h$, floats in equilibrium (i) with its vertex downwards and its base in the upper surface, (ii) with its vertex upwards and its base in the surface between the second and third liquids. Show that the ratios of the densities of the cone and liquids are 31 : 30 : 33 : 36.

34. A rod of cork, 8 inches long, and a rod of lignum vitæ, 4 inches long, are joined together so as to form a rod of uniform small section. When the rod floats in water it is found to be capable of resting in equilibrium with part of the cork above the surface, and with its axis inclined at any angle to the vertical. Given that the specific gravity of cork is ·24, find the specific gravity of lignum vitæ.

35. A cube (specific gravity 1·2) floats in a liquid whose specific gravity is 1·3 with four faces vertical. Water is poured on till it just reaches the level of the upper surface of the cube. Find the centre of pressure of one of the vertical faces of the cube.

36. A triangular lamina ABC, right-angled at C, floats in a liquid of 8/3 of its own density, hinged freely at C to a point fixed below the surface, and with AB entirely out of the liquid. If CA makes an angle of 30° with the horizon, and CB is bisected by the surface of the liquid, prove that the lengths of CA and CB are to one another as $2 : \sqrt{3}$.

37. ABCD is a lamina in the form of a parallelogram, in which the angle BAD$=60°$, and the diagonal BD is perpendicular to the side AD. The

lamina is hinged freely at A to a fixed point, and floats partly immersed in a liquid, so that AB is horizontal, and that the side AD is divided by the surface of the liquid at F, so that $2AF = FD$. Find the ratio of the densities of the liquid and the lamina, and show that the pressure on the hinge is 1/16 of the weight of the lamina.

38. A square lamina of specific gravity $(25 - 7\sqrt{3})/18$ floats in water with the vertex A hinged to a point in the surface and the side AB out of the water. Find the inclination of this side to the vertical.

39. Prove that the time taken by a cylindrical cistern to empty itself by a small hole in the bottom, no water being supplied during the emptying, is double the time in which the same quantity of water would flow from the same cistern through the same orifice, the cistern being kept full of water supplied at the top.

40. A fine tube, consisting of a horizontal part connecting two vertical branches, is partly filled with water and made to rotate uniformly round one of the vertical branches; find the difference of the level of the water in the two branches.

If the length of the horizontal part is 6 inches, and the rotation is at the rate of 4 turns a second, what is the difference of level? [$g = 32$].

41. A Y-tube is inverted, and each leg is dipped into a different liquid. Air is then sucked from its stem, and the liquids rise to vertical heights of 17 and 15 centimetres respectively. Compare the specific gravities of the two.

If the heavier of the two liquids is water, find the pressure of air in the stem if the height of the barometer is 75 centimetres. [Specific gravity of mercury is 13·6].

42. A cylindrical diving-bell, 240 centimetres high, is sunk to the bottom of a water reservoir 516 centimetres deep. Find the height to which the water will rise in it, the water barometer reading 1036 centimetres.

43. A cylindrical diving-bell is lowered in water to such a depth that the confined air occupies 2/3 of the internal volume of the bell. If half as much air again is pumped into the bell, how much farther must the bell descend before it becomes half full of water?

44. A cylinder of height 5 feet, with its axis vertical if full of air at atmospheric pressure, and is closed at the top by a tightly-fitting piston of mass 30 lbs. If the piston sinks 2 feet under its own weight, determine the pressure that must be applied to the piston in order to force it down through an additional distance of 2 feet.

45. The height of a cylindrical diving-bell is 12 feet, and the area of its base is 64 square feet. The water barometer stands at 35 feet 5 inches, and the temperature of the water is the same as that of the atmosphere. Find the least weight of the bell that it may sink to rest upon the bottom of a harbour 56 feet 7 inches deep, assuming that a cubic foot of water weighs 1000 oz.

46. A conical diving-bell, of which the axis is 16 feet, is let down into water, and it is found that when the vertex is $33\frac{2}{3}$ feet below the surface

the water has risen within the bell to a height of 4 feet. Find the height of the water barometer.

47. At the top of the closed tube of a barometer a portion of air remains which occupies one inch of the tube, and has a pressure of 2 inches of mercury, that is, 1/15 of an atmosphere. The cistern of the barometer has a sectional area four times that of the tube. If a solid cylinder of height 63/4 inches, and sectional area equal to that of the tube, and with a density equal to half that of mercury be now allowed to float in the cistern, what will be the pressure of the confined air?

48. A tube of length l, greater than the barometric height, is filled with mercury and inverted over a cistern of mercury, the level of the fluid in the tube being at a height b above that of the cistern. A mass m of gas is introduced into the tube, and the mercury sinks to the height h, the level of the cistern being kept constant. The tube is then emptied and refilled with mercury, and a mass m' of another gas being introduced, the mercury sinks to a height h'. The temperature and atmospheric pressure being the same throughout, find the ratio of the densities of the gases at the same temperature and pressure.

49. At a place where the height of the water barometer is 34 feet, and the temperature of the air is 0° C., a diving-bell, whose capacity is 84 cubic feet, and which is originally full of air at atmospheric pressure and temperature, is lowered into water at 21° C. until its lower edge is 17 feet below the surface. How many cubic feet of air at atmospheric pressure and temperature must be pumped into the bell in order that when the contained air has acquired the temperature of the water it may just fill the bell?

[The coefficient of expansion for gases at constant pressure is 1/273.]

50. A piston which moves freely but air-tight in a smooth cylinder, is placed at the middle of the cylinder, and the ends are then closed. On placing the cylinder vertical, the distance of the piston from the top is to its origin at distance as $\sqrt{2}:1$. On raising the air in the two parts of the cylinder to the absolute temperatures t_1 and t_2, the piston goes back to the middle of the cylinder. Show that the original temperature of the air in the cylinder was $t_1 \sim t_2$.

EXAMINATION PAPERS.

(The Answers are given on pages 339, 340.)

I. SCIENCE AND ART—THEORETICAL MECHANICS. FLUIDS, 1892.

Second Stage or Advanced Examination.

[You are not permitted to attempt more than *eight* questions. You may select them from any part of the paper.]

1. Define moment of inertia. Find the moment of inertia of a circular lamina about an axis at right angles to its plane and passing through its centre. (20.)

2. What are the rectangular components of a force?

A cylinder is put under water with its axis horizontal; a very small portion of the area of its curved surface is taken; find the horizontal and vertical components of the pressure of the water on that area. Find also numerical results in the following case: — The radius of the cylinder is 2 ft.; the axis is 4 ft. below the surface; find the horizontal and vertical components of the pressure on an area of 0·1 square inch, at a depth of 3 ft. below the surface.

N.B.—Take a cubic inch of water to weigh 250 grains. (20).

3. A particle whose mass is 5 lbs. moves at the rate of 20 ft. a second; express its kinetic energy in foot-poundals. If it moves over a distance of 30 ft. against a constant resistance (R), and its velocity is thereby reduced to 15 ft. a second, find R in poundals. (25.)

4. A particle whose mass is 10 lbs. is constrained to move in a horizontal circle by a string 5 ft. long fastened to a fixed point; if at any instant the tension of the string is 98 poundals, find the velocity of the particle, and its angular velocity about the fixed point. (25.)

5. Define the centre of pressure of a plane area.

ABCD is a rectangle, whose plane is vertical; draw the diagonal AC; if the rectangle is immersed with the edge AB in the surface of water, find from first principles the centre of pressure of the triangle ACD. (25).

6. A hollow sphere is under internal fluid pressure; putting out of the question the weight of the fluid, find the resultant pressure on a given half of the surface. (25.)

7. Explain what is meant by the absolute zero of temperature. If the absolute zero on the Centigrade scale is $-273°$, what is it on Fahrenheit's scale?

What experimental facts and what suppositions are implied in the determination of the absolute zero? (20.)

8. Show how to find the work done by a gas that expands at a constant temperature.

The quantity of air which occupies a volume of 50 cubic feet under a pressure of 15 lbs. per square inch is allowed to expand at a constant temperature to 60 cubic feet; find the number of foot-pounds of work done (Hyp. log. $1 \cdot 2 = 0 \cdot 1823216$). (30.)

9. Describe the siphon-manometer, and explain how it may be graduated.

If a siphon-manometer is to read pressures up to three atmospheres, what quantity of mercury will be required, the area of the section of the tube being a quarter of a square inch? (20.)

10. With given materials, show that the rise (due to Capillary Action), of a liquid in a circular tube equals the rise between two plates, whose distance equals the radius of the tube. (20.)

11. A cylindrical vessel is 10 ft. high, and its base, which is horizontal, has a radius of 1 ft.; it is filled with water, and a small hole is made in the bottom. Compare the rate of descent of the surface when the hole is first opened with the rate when the vessel is half empty. If the vessel is half emptied in 20 minutes what is the effective area of the hole? (30.)

12. Define a simple harmonic motion, and show how it can be represented (*a*) graphically by a diagram, (*b*) by a formula. Define the amplitude, epoch, period, and phase of a simple harmonic motion. (20.)

II. SCIENCE AND ART—FLUIDS, 1893.

Second Stage or Advanced Examination.

[You are not permitted to attempt more than *eight* questions.]

1. Define moment of inertia.

If the moment of inertia of a body with reference to an axis passing through the centre of gravity is known, how can the moment of inertia be found with respect to a parallel axis? (20.)

2. Define the rectangular components of a force.

A conical vessel is full of water; the half angle at its vertex is 30°; find the pressure at a point of the surface of the cone 2 feet below the surface of the water, and finds its horizontal and vertical components. N.B.—A cubic inch of water may be taken to weigh 250 grains. (20.)

3. Define angular velocity.

A particle, whose mass is 3 lbs., moves uniformly in a circle; it describes the circumference 42 times a minute; find its angular velocity about the centre; and, if the radius is 14 ft., find its kinetic energy. (20.)

4. State and prove a rule for finding the whole pressure of a liquid on a body immersed in it.

A sphere, whose radius is 2 ft., has its centre 3 ft. below the surface of water; find the whole pressure of the water on it; find also the resultant pressure of the water on it. (20.)

5. The formula for the determination of the metacentre being $HM . V = Ak^2$, state the meaning of each term of the formula.

Given that the moment of inertia of a circular lamina about a diameter is $Mr^2 \div 4$, find the relation between the length and diameter of a cylinder of cork, when it floats with its axis vertical; the specific gravity of cork being 0·25. (25.)

6. Find the tension of a thin cylinder under internal fluid pressure.

If the internal diameter of the tube is 8 in., the thickness of the material 0·1 in., and its tenacity 20,000 lbs. per square inch, find the bursting pressure in pounds per square inch. (25.)

7. The relation between the volume, pressure, and temperature of a gas being $VP = CT$, state the meaning of each letter of the formula.

The coefficient of expansion being 0·003665, state exactly what it means. If a given quantity of gas expands under a constant pressure, in consequence of its temperature being raised from 40° C. to 41° C., find what ratio the increase of volume bears to its volume at 40° C. (25.)

8. A cylinder, partly filled with water, turns with a constant angular velocity round its axis, which is vertical; find the form of the surface of the water.

If the radius of the cylinder is one foot, and it turns three times a second, find the pressure per square inch at a point of the cylinder 6 in. below the lowest point of the surface of the water. (30.)

9. Describe the Hydrometer of variable immersion.

A hydrometer is put into a liquid whose specific gravity is 0·9, and a mark (A) is placed at the point of the stem which is on the surface of the liquid; it is next placed in a liquid whose specific gravity is 0·8, and a mark (B) is placed at the point of the stem which is now on the surface of the liquid; it is found that B is 4·3 in. above A; find the point of the stem that will be on the surface of a liquid whose specific gravity is 0·86 when the hydrometer floats in it. (25.)

10. If a small quantity of water is introduced into a barometer—*e.g.* such a quantity as would occupy a quarter of an inch of the length of the tube—what effect will it have on the height of the mercurial column? If the pressure of the atmosphere continues constant, what will be the effect of raising the temperature of the mercury and water? (25.)

11. A vessel of water 9 ft. high is kept constantly full of water; a hole, whose area is a quarter of a square inch, is made in its bottom. What would be the theoretical outflow in one hour? How would the actual outflow differ from the theoretical outflow? (25.)

12. A straight tube is filled with air; a disc which nearly fills a section of the tube oscillates rapidly through a small distance within the tube. Describe the disturbance propagated along the tube, particularly the condensation and rarefaction set up. Define the phase of a vibration and the length of a wave. (25.)

III. SCIENCE AND ART—FLUIDS.

Honours Examination.

[The following questions, which are taken from the Honours papers in Fluids for the years 1892 and 1893, can be answered by the principles explained in this book, and without the use of the Calculus. The papers each contain twelve questions, of which not more than *eight* are to be attempted.]

A.—1892.

1. If the unit of work be 2520 foot-pounds, the unit of force the weight of a mass of 784 lbs., and the unit of time 3 seconds, find the units of mass and distance. $(g=32)$

2. Draw an equilateral triangle ABC, with AB horizontal and C downwards; let AC and BC represent equal threads supporting at C a particle whose mass is 12 lbs.; if the string BC is cut, find the tension of AC (*a*) immediately before, (*b*) immediately after the severance. Find also the tension at the instant the mass is vertically below A.

3. There are three liquids A, B, C; a hydrometer of variable immersion is placed in them successively; it floats with 2 inches of its stem out of A, with 3 inches of its stem out of B, and with 4 inches of its stem out of C; the specific gravity of A is 0·8, that of B is 0·85. Find from first principles the specific gravity of C.

4. A cone is held under water with the highest straight line of its curved surface horizontal; find the magnitude and direction of the resultant fluid pressure on its curved surface.

Explain the result obtained for the direction of the resultant when the depth at which the body is immersed becomes very great.

5. Apply the formula $V.\mathrm{HM}=Ak^2$ to find the condition that a cylinder, whose specific gravity is 0·8, may float in water in stable equilibrium with its axis vertical.

6. Two barometers stand on the level of the sea; the one, in which the vacuum is perfect, reads h; the other reads k, and has air in the vacuum space, the length of which is l; if the acceleration of gravity were changed from g to g', show what the effect would be on the reading of the barometers.

Obtain numerical results when h, k, and l are 30, 28, and 6 inches respectively, and g and g' are 32 and 24.

7. A column of water, a quarter of a square foot in section, descends on a horizontal inelastic area at the rate of 3000 gallons a minute; the water runs freely off the area. Find the pressure on the area in pounds per square foot. $(g=32)$

B.—1893.

1. Define angular velocity.

ABC is a triangle, and a point moves with a given velocity from B to C; compare its angular velocity about A, when it is at B, with what it will be when it reaches C.

2. Two particles are acted on by constant forces through distances in the ratio of 27 to 14; they thereby acquire kinetic energies in the ratio of 6 to 7; their masses are in the ratio of 7 to 3; find the ratio of the forces and the ratio of the accelerations due to the forces.

3. Investigate a formula for finding by the balance the specific gravity of an insoluble solid lighter than water.

A piece of cork (sp. gr. 0·25) weighs 3 oz.; it is fastened to a piece of brass (sp. gr. 8). Find the weight of the brass when the weight of the whole in water is 12 oz.

4. A cone floats with its axis vertical and vertex downward; find the position of the metacentre, and, for a given vertical angle, find the specific gravity when the equilibrium is just not unstable.

IV.—GLASGOW UNIVERSITY.

PRELIMINARY EXAMINATIONS IN ARTS AND SCIENCE—OCTOBER, 1892.

DYNAMICS.

[Only the questions in Hydrostatics and Pneumatics are given here.]

1. Explain the terms—*pressure at a point in a fluid, total pressure, resultant pressure.*

A cubical box, one foot high, is filled to a depth of 6 inches with mercury, and is then filled up with water. Find the total pressure on the base and on one of the sides of the box.

[Specific gravity of mercury is 13·6. Mass of a cubic foot of water is 62½ lbs.]

2. State the principle of Archimedes.

A solid weighs 30 grammes in water, and 40 grammes in a liquid whose specific gravity is ·8. What is the volume of the body in cubic centimetres, and what is its mass in grammes?

3. State Boyle's law, and show how it may be verified for common air, *either* for pressures greater than atmospheric pressure, *or* for pressures less than atmospheric pressure.

4. Describe briefly the construction of the mercurial barometer in its simplest form.

Using the data of Question 1, calculate the atmospheric pressure in pounds weight per square inch when the mercury barometer stands at a height of 30 inches.

V.—GLASGOW UNIVERSITY.

M.A. DEGREE EXAMINATION.—APRIL, 1893.

Natural Philosophy.

[Only the questions in Hydrostatics and Pneumatics are given here.]

1. Distinguish between a solid and a fluid, and between a liquid and a gas. Give a criterion of a perfect fluid. How would you class such a substance as pitch or shoemaker's wax? State carefully the reason for your answer.

2. Define pressure at a point in a fluid, and prove that in a perfect fluid the pressure at each point is the same in all directions.

Prove that if two columns of different liquids equilibrate one another in a U-tube, the heights above the common surface are inversely as the densities of the liquids. Describe a simple apparatus for comparing by this means the densities of two liquids.

3. State the principle of Archimedes.

State the conditions of equilibrium of a floating body, (1) when the body is wholly immersed, (2) when it is only partially immersed. Define the terms centre of buoyancy, metacentre, metacentric height, and show how the righting moment depends on the angle of heel and the latter quantity.

4. A balloon, containing 10,000 cubic feet of gas at 15° C., and pressure of 75·3 centimetres of mercury, on rising is cooled to 3° C., and the pressure is changed to 64·7 centimetres. Find the volume of the gas.

VI.—EDINBURGH UNIVERSITY.

PRELIMINARY EXAMINATIONS IN ARTS AND SCIENCE—OCTOBER, 1892.

Dynamics.

[Only the questions on Hydrostatics and Pneumatics are given here.]

1. Define specific gravity.

A mass of 12 lbs. of a certain substance is found to weigh 9 lbs. in a liquid whose specific gravity is 1/3. What is the specific gravity of the substance?

2. Find the pressure on a lock gate, which is vertical and 10 feet broad, and against which water rises 4 feet. [A cubic foot of water weighs 62·5 lbs.]

3. State Boyle's Law in the form of a relation between the pressure and density of a gas. Describe the method by which the law is proved experimentally for pressures less than an atmosphere.

VII.—EDINBURGH UNIVERSITY.

M.A. DEGREE EXAMINATION—APRIL, 1893.

NATURAL PHILOSOPHY.

[Only the questions on Hydrostatics and Pneumatics are given here.]

1. Prove that the resultant pressure acting on the surface of a body wholly or partly immersed in a fluid at rest is equal to the weight of fluid displaced.

The apparent weight of a body tested by a spring balance is 3 lbs. when the body is half immersed in water, and 1 lb. when three-quarters immersed. Find the weight and specific gravity of the body.

2. Prove that the pressure in a heavy liquid at rest is the same at all points of a horizontal plane.

A tank, 4 feet square and 10 feet deep, is filled with water. Find the pressure on the bottom of the tank and on one side. [The weight of a cubic foot of water is $62\frac{1}{2}$ lbs.]

3. Describe the action of the force pump.

If the cross section of the piston has an area of 5 square inches, with what pressure must it be forced down to the capable of raising water to an elevation of 100 feet?

4. State the relation between the pressure, volume, and temperature of a given mass of gas.

If the pressure is increased by 1/10, and the temperature raised from 50° C. to 100° C., in what ratio will the volume be altered?

ANSWERS TO EXAMPLES.

I. (Pages 16–18.)

1. (i) 100. (ii) 3700. (iii) 387000. (iv) ·23. (v) 6320. **2.** (i) 3200. (ii) 45. (iii) ·00373. **3.** (i) 10000. (ii) 1156. (iii) 90000. (iv) 5·29. **4.** (i) 3375000. (ii) ·054872. (iii) 12·167. **5.** (i) $18\frac{1}{3}$. (ii) $12\frac{4}{7}$. (iii) $56\frac{4}{7}$. (iv) 6, 6, 8, 8, 12, 12. **6.** (i) 64. (ii) 192. (iii) $169\frac{5}{7}$. (iv) $471\frac{3}{7}$. (v) $268\frac{4}{21}$. **7.** 286. **8.** $490\frac{2}{7}$.

II. (Page 24.)

1. 18. $11\frac{1}{4}$ ft./sec. **2.** 112 units. **3.** $23333\frac{1}{3}$ ft./sec. **4.** 20 lbs. **5.** 40. **6.** 160. **7.** 2430000 foot-poundals. $90\sqrt{3}, =156$ ft./sec. nearly. **8.** (i) 100 dynes. (ii) 5000 dynes. **9.** (i) $273777\frac{7}{9}$. (ii) $1003851\frac{23}{27}$. **10.** 594 feet. 6 feet. **11.** 24. **12.** 48000 poundals. **13.** $4\frac{5}{11}$.

III. (Pages 27–29.)

A.

1. 9600 units. $76\frac{4}{5}$ ft./sec. **2.** $40\sqrt{65}, =322·5$ ft./sec. nearly. **3.** 3303384. **4.** (i) 3 : 5. (ii) 2 : 3. **5.** (i) 11/12. (ii) $4\frac{13}{27}$. **6.** $2111\frac{1}{2}$ nearly. **7.** 46875000. **8.** 1612800. **9.** 125/512 of a foot. **10.** 125. 12. **12.** $2\frac{1}{27}$. **13.** $1\frac{5}{14}$ tons weight. **14.** 952. **15.** 5950 lbwt. $3781\frac{1}{4}$ yards. **16.** 160. $238\frac{4}{15}$.

B.

17. 250440 lbwt. nearly. **18.** ·78 of a second. 217 : 162. **19.** 437·5. **20.** $\sqrt{2Wg/m}$ ft./sec. **21.** (i) The momenta are equal. The kinetic energy of m is to that of m' as $m' : m$. (ii) The kinetic energies are equal. The momentum of m is to that of m' as $\sqrt{m} : \sqrt{m'}$. **23.** 2000, taking $6\frac{1}{4}$ gallons to be equal to a cubic foot. **24.** $70246\frac{2}{5}$ lbs. $3\frac{3}{14}$ feet. **25.** $18\frac{3}{4}$ foot-pounds.

IV. (Pages 35–37.)

A.

1. 11200 : 1. **2.** 9800. **3.** 48 : 1. **4.** 75 : 1024. 81 : 16. **5.** $718\frac{3}{4}$. **6.** 9350 lbs. **7.** 96/275 of a square inch. **8.** 9/275. **9.** 5/12 of a square centimetre. **10.** ·79 of a square centimetre nearly. **11.** ·88. **12.** 94 nearly. **13.** 25/26. **14.** ·875. **15.** 1737 parts of gold to 221 parts of silver. **16.** Density of air in A is five times that of air in B.

B.

17. 1 : 3. **18.** 7519 : 10585 : 14935. **19.** 1075 : 1825 : 3139. **20.** (i) Equal volumes. (ii) $s : s'$. **21.** (i) $s' : s$. (ii) Equal weights. **22.** 16 lbs. Sp. gr. of A is $1\frac{3}{5}$; sp. gr. of B is 4/5. **23.** kk'.

V. (PAGES 57–61.)

A.

3. $\sqrt{64+32\sqrt{2}}$ units acting along the line bisecting the angle between the forces 6 and 7. **4.** $10\sqrt{5}$ lbwt. inclined to the vertical at an angle whose tangent is 1/2. **5.** 43·3 and 25 units respectively. 48·3 and 12·9 units respectively. **6.** 65 units. **7.** 50 lbwt.; $50\sqrt{3}$ lbwt. **8.** Inclined at an angle of 30° to the vertical. **9.** Half the weight of the ladder. **10.** Half the weight of the board. **11.** 3·03 inches. **12.** 2/5 of a foot. 11/40 of a foot from the base. **13.** 1/30 of a foot from the centre of the board. **14.** 1/9 of the side of the square from the centre of the square. **15.** $5\sqrt{3}/3$ inches. **16.** $7\frac{1}{3}$ inches and $8\frac{1}{3}$ inches respectively. **17.** $2\frac{6}{7}$ feet from the end referred to. **18.** $2\frac{2}{3}$ feet. **19.** 12 lbs. At its middle point. **20.** At its middle point.

B.

21. This is equivalent to the geometrical problem—to construct a right-angled triangle, given the hypotenuse and one of the sides containing the right angle. **22.** The two forces must act along lines which are equally inclined to the line of action of the given force. The problem thus reduces itself to a well-known geometrical problem. **23.** The straight line bisecting at right angles the line which represents the resultant. **24.** 26·6 units; 21·8 units. **27.** Use the triangle of forces. If AB and CD, produced if necessary, meet in O, the resultant acts through O parallel to HK. If AB and CD are parallel, O is at infinity and HK is zero. The two given forces then form a couple; and it follows from the construction for the resultant of two unlike parallel forces which are very nearly equal that a couple may be regarded as an infinitesimal force acting at infinity. **31.** Let P and Q denote respectively two of the forces, and R, R the other two, this latter pair forming a couple. Replace this couple by a couple of equal moment and of the same sense, having Q, Q as forces. The couple formed by Q, Q may be placed so that one of these forces shall be opposite to the given force Q. The latter force will thus be balanced, so that the given system will be equivalent to the given force P, and the unbalanced force Q of the couple. The resultant of these two forces will be the resultant of the given system of four forces. For the second part of the question produce DA to E, making AE = 3DA/2; draw EF perpendicular to DE; and produce CA indefinitely, cutting EF in F. The resultant of the given forces is equivalent to the resultant of two forces, 10 along FE and 20 along AF, and is very nearly 14·7. **32.** Produce DC to E, making

CE = DC; produce AD to F, making DF = 3AD. Then the given system is equivalent to forces represented by DE and DF. **35.** $a/12$. **38.** A circle. **40.** In the line drawn from the first position of the centre of gravity perpendicular to the base, at a distance 1/12 of the length of this line from the first position of the centre of gravity. **46.** The force at C is $W\sqrt{3}/2$, and acts horizontally; the force at B is $W\sqrt{7}/2$, and acts in a direction making with the vertical an angle whose tangent is $\sqrt{3}/2$.

VI. (Pages 103–105.)

1. (1) 625. (2) 2187·5. (3) 165000. **2.** (1) 2785. (2) 4347·5. (3) 167160. **3.** 84 lbs. per square inch. **4.** 7 : 3. **5.** $1\frac{3}{7}$ feet. **6.** Depth of A is 16 feet; depth of B is 4 feet. **7.** $1777\frac{7}{9}$ lbs. **8.** $1\frac{1}{2}$. **9.** (i) 3·17. (ii) 1·69. **10.** 125/144 lbwt. per square inch. **11.** ·133. **12.** ·52. **13.** ·289 lbwt. **14.** 10 inches. **15.** A volume which occupies $13\frac{1}{3}$ inches of the tube. **16.** 1·252. **17.** $4\frac{2}{21}$ inches above the base of A. **18.** Vertical component is 25/384, = ·0651 lbwt.; horizontal component is $25\sqrt{3}/384$, = ·1128 lbwt. **19.** Intensity of pressure is ·03617 lbwt. per square inch. Pressure on 1/16 of a square inch is ·00226 lbwt. The vertical and horizontal components of this pressure are ·0015 lbwt. and ·00168 lbwt. respectively. **20.** $3\frac{17}{36}$. Pressure on 1/10 of a square inch is 25/72 lbwt.; and the vertical and horizontal components of this pressure are $25/36\sqrt{13}$ lbwt. and $25/24\sqrt{13}$ lbwt. respectively. **21.** $4\frac{223}{288}$. Pressure on 1/20 of a square inch is 275/1152 lbwt.; and the vertical and horizontal components of this pressure are $275/384\sqrt{13}$ lbwt. and $275/576\sqrt{13}$ lbwt. respectively.

VII. (Pages 113–115.)

1. 58 lbs. per unit of area. **2.** 36750 lbwt. **3.** 24000, 300000. **4.** Pressure on bottom 1687·5; on one of the sides 843·75. **5.** 21127·5; 20283·75. **6.** 4200. **7.** 2720. **8.** Pressure on bottom is the weight of 54·4 cwts.; on one of the sides is the weight of 27·2 cwts. **9.** (i) 9350. (ii) 216800. **10.** Pressure on base 327·6; on sides 540, 819, 981 respectively. **11.** 27·96, 23·09, 16·16 respectively. **12.** Pressure on base 58·33; on one of the sides 28·47. **13.** Pressure on base 75·52; on one of the sides 36·89. **14.** $3\sqrt{3}$ inches. **15.** 40. **17.** (i) As $h : r$; (ii) as $2h : r$, when h is the height of the cylinder, and r the radius of the base.

VIII. (Page 123.)

1. 1500 lbwt. **3.** (i) $7a/12$. (ii) $17a/20$. (iii) $25a/36$. (iv) $15a/28$. **4.** (i) $7a\sqrt{2}/12$. (ii) $7a\sqrt{2}/12$. (iii) $11a\sqrt{2}/16$. (iv) $11a\sqrt{2}/16$. (v) $5a\sqrt{2}/8$. **5.** (i) $7h/12$. (ii) $7h/12$. (iii) $17h/20$. (iv) $7h/12$. (v) $55h/76$. (vi) $31h/44$. **6.** $17a/8$.

IX. (Pages 135–139.)

A.

1. 39484·375. **2.** 2812·5. **3.** 2604·17, 5208·34. **4.** As 2 : 1 : 3. **5.** 6015·625. **6.** 521 nearly. **7.** *Total* pressure is 2322000 lbwt.; *resultant* pressure is 1032 ozwt. **8.** 703125. At the centre of pressure which is at a depth of 100 feet. **9.** 468750. A line perpendicular to the side through the centre of pressure which is at a depth of 66$\frac{2}{3}$ feet. **10.** 12187·5. The point in the vertical line bisecting the area of the gate at a height of 3$\frac{4}{13}$ feet above the base of the lock. **11.** (i) 132·8 nearly. (ii) 188·7. **12.** (i) 407·6. (ii) 8·2. (iii) 247·2. (iv) 239. **13.** 531 nearly. 239·5 nearly. **14.** It will rise. **15.** (i) 100 grammes. (ii) 37 grammes. **16.** 210·93 grains nearly. **17.** 213·04 grains nearly. **18.** 32000. **19.** Not less than 41288·2 litres.

B.

20. 60°. The depth of water must not be less than $r\sqrt{3}$, and the height of the cylinder must not be less than $2r\sqrt{3}$, where r is the radius of the base. **21.** AP = 3AB/5. **22.** If h denotes the length of the perpendicular from the vertex on the base, the lines required are at distances $h/\sqrt[3]{3}$ and $h\sqrt[3]{2}/\sqrt{3}$ from the vertex. **23.** 50 inches; $10\sqrt{5}(\sqrt{3}-\sqrt{2})$ inches. **25.** (iii) The weight of the liquid. **26.** $2w(4-\sqrt[3]{4})$. **29.** 3/4 of the depth of C. **31.** The resultant horizontal pressure is $rh^2w\sqrt{2}/6$, and the resultant vertical pressure is $\pi r^2hw/12$. The resultant pressure is $wrh\sqrt{h^2/18+\pi^2r^2/144}$. **32.** $\pi r^3w\sqrt{13}/3$. **34.** That the resultant horizontal pressure is zero is an immediate deduction from the principle of Archimedes. The same thing may be proved by dividing the imaginary surface into two parts by any vertical plane, and applying the method of Art. 76 to prove that the resultant horizontal pressures on the two parts are equal and opposite forces. It will at once be seen that the horizontal pressure on one of the parts is equal to the stress upon the vertical plane; and the above result immediately follows. Also, it follows that the resultant pressure on the imaginary surface must be a vertical force; and since the resultant pressure balances the weight of the fluid inclosed in the surface, it must be equal to this weight, and must act upwards through the centre of this fluid. **35.** 1249 lbwt. **37.** The resultant horizontal pressure is $\pi r^2hcw/\sqrt{r^2+h^2}$, and the resultant vertical pressure is $\pi r^2w(cr/\sqrt{r^2+h^2}+h/3)$. The resultant pressure is the square root of the sum of the squares of these two expressions.

X. (Pages 151–153.)

1. 3; ·75. **2.** 161·94 lbs. 2·59. **3.** 3·6. **4.** ·8. **5.** ·2. **6.** ·25. **7.** 5/6. **8.** 175 grains. **9.** 27/34. **10.** 1$\frac{2}{3}$ oz. **11.** 2. **12.** 4. **13.** 8. **14.** As 10 : 13. **15.** 21. **16.** 1·125. **17.** $11\cdot\dot{1}$ grammes per cubic centimetre. **18.** 7$\frac{2}{13}$. **19.** 25 grammes.

20. 13/18 of the whole volume of the shell. **21.** $1\frac{1}{11}$. **22.** 18/19. **23.** $(b-as)/(1-s)$ grammes; $(b-a)/(1-s)$ cubic centimetres. **24.** As $W_0 - W_1 : W_0 - W_2 : W_0 - W_3 : \ldots\ldots$ **25.** No. **26.** 1451·91 grains.

XI. (Pages 161–165.)

A.

1. 3/4. 140·625 lbwt. **2.** (i) 1·4 feet. (ii) 147·3 lbwt. (iii) 147·3 lbwt. **3.** 7 inches. 6/7. **4.** 1·1. **5.** 7/25. **6.** 36·42 cubic centimetres. 7·55. **7.** $16\frac{2}{3}$ inches. **8.** $6\frac{1}{13}$ inches. **9.** 3/4; 5. **10.** It would float. **11.** 54/133 of a stone. **12.** 354·3 lbwt. **13.** 50 cubic centimetres. **14.** 6 lbwt. **15.** No. **16.** 1159/1375. **17.** It sinks. **18.** There will be a decrease. **19.** 7·3. **20.** 65/126. **21.** $17\frac{1}{3}$ grammes. $26\frac{2}{3}$ grammes. **22.** 407 lbwt. **23.** 33/38 of a cwt. 146 lbs. nearly. **24.** $1\frac{1}{10}$. 13/30. **25.** $6\frac{3}{38}$ lbs. **26.** $62\frac{1}{2}$ lbwt. **27.** 75 lbwt. **28.** 4 lbwt. **29.** 45·1 lbwt. nearly. **30.** 234·34 grains.

B.

31. ·73 of an inch. **34.** 5 lbs. 1·25. **35.** Let a denote the external edge of the box, k the thickness of the boards, and s the specific gravity of the wood. Then we find that the depth when water-tight is approximately $6ks$, and when not water-tight $a(6s-1)/4$. In the numerical example the depth when water-tight is $1\frac{1}{2}$ inches, and that when not water-tight the box floats with the upper horizontal face just not immersed. **37.** As AC : BC cos BCA. **38.** $(A'-A)hs/A's'$. **39.** $10\pi r^3/3$. **40.** The volumes of the lower and upper liquids displaced are $a^3(1-m)/(n-m)$ and $a^3(n-1)/(n-m)$ respectively, the ratio of the first to the second being equal to $(1-m)/(n-1)$. **41.** The condition is $2m=n+1$. **43.** $2(a^2+ab+b^2)/(5a^2+2ab+2b^2)$. **44.** $\sqrt[3]{2s-1} : \sqrt[3]{2s}$. **45.** At an angle of 60° to the vertical.

XII. (Pages 172–173.)

1. W.CG. $\sin\theta$. (i) 150. (ii) $150\sqrt{2}, =212$. (iii) $150\sqrt{3}, =260$. (iv) 300. **2.** 500. **3.** 385.

XIII. (Pages 179–180.)

1. 10° 12′. **2.** 5°. **10.** $r/\sqrt{s(1-s)}$, where r denotes the radius and s the ratio of the specific gravity of the material of the cylinder to the specific gravity of the liquid. **12.** Stable or unstable according as $b/c >< \sqrt{6\sigma(\rho-\sigma)/\rho^2}$.

XIV. (Pages 194–196.)

1. The length of the column of mercury would be longer than if it were vertical, so that the reading would be increased. **2.** No. **4.** 41130 lbwt.

5. 23·7. **6.** 28·02. 1·87 atmospheres. **7.** 135 feet. **8.** 875. **9.** 2160 lbwt. 1080 lbwt. **10.** 212$\frac{1}{7}$ lbwt. **11.** 1476 lbwt. nearly. **12.** 424$\frac{2}{7}$ lbwt. **13.** 10·52 lbwt. **14.** 1·167.

XV. (Pages 207–209.)

1. 33$\frac{3}{4}$ lbwt. **2.** 82944 grains. **3.** 290$\frac{5}{8}$ cubic centimetres. **4.** 443$\frac{1}{13}$ grains. **5.** 14/15 of a gramme. **6.** 30 lbwt. per square inch. **7.** 6$\frac{2}{3}$ and 33$\frac{1}{3}$ lbwt. per square inch respectively. **8.** 4$\frac{1}{2}$ inches. **9.** 74·8. **10.** In the tube a pressure of 25 inches of mercury, in the receiver a pressure of 21 inches of mercury. **11.** 3$\frac{1}{5}$ atmospheres. **12.** 3 inches. **13.** 30·25 inches. **14.** 30·5 inches. **15.** 3 inches. **16.** 7/8 of a cubic inch. **17.** 4 cubic inches. **18.** 14·2. **19.** 26·7 cubic inches. **20.** 19·7 inches. 23·15 inches.

XVI. (Pages 218–220.)

A.

1. Height of the barometer is 30 inches; pressure of compressed air is that of 42 inches of mercury. **2.** 7·16 inches. **3.** 3/7 of a foot. **4.** 4·4 feet nearly. 257$\frac{1}{7}$ cubic feet. **5.** 240 million ergs. **6.** 46352. **7.** 79851. **8.** 30761. **9.** 31923, taking an atmosphere to be a pressure of 14·8 lbwt. per square inch.

B.

10. $h(2n-1)/n(n-1)$. **11.** $d(n+d)/(n-d)$ centimetres. **12.** Height of barometer is $(4b-a)$ inches; pressure of compressed air is that of $4b$ inches of mercury. **13.** $a(p'-p)/d(p'v'-pv)$. **14.** The weight of the piston is, of course, neglected. aAp hyp. log. $(1+c/a)$. 18326 foot-pounds. Notice that, since the initial and final volumes are given, the area of the piston is not required. **15.** If h_1 and k_1 denote what would be the heights of the perfect and faulty barometers respectively, then $h_1=gh/g_1$, and k_1 is given by the equation $(a+k-k_1)(gh-g_1k_1)=g(h-k)a$. In the numerical example $h_1=40$, $k_1=32$.

XVII. (Pages 244–245.)

3. A pressure of 31·59 inches of mercury. **4.** 14°·4 C. nearly. **5.** 546° C. **6.** 1132000, 112250 grains. **7.** (i) 7808. (ii) 7407. (iii) 7286. **8.** The mercury will divide the tube into two parts whose lengths are in the ratio of 149 to 144. **10.** 5259. **11.** 1164. **12.** 16·9. 16·7. **13.** A pressure of 119·5 inches of mercury.

XVIII. (Pages 259–260.)

1. (i) 888$\frac{8}{9}$ lbwt. per square inch. (ii) 9/200 of an inch. (iii) 20 inches. **2.** 18000 lbwt. per square inch. **3.** 1100 lbwt. per square inch.

XIX. (Pages 275–276.)

[Take $g=32$ ft./sec.^{2}, and $\pi=22/7$.]

1. 6 feet from the side of the cylinder. **3.** 200. **4.** $8\frac{1}{3}$. **5.** As 1 : 4800. **6.** (i) 7/891 ft./sec. (ii) 7/1188 ft./sec. (iii) 7/1782 ft./sec. (iv) 19·88. (v) 67·89. **7.** 1339 ft./sec.

XX. (Pages 283–284.)

1. $90/\pi, =28\frac{7}{11}$. 30 ft./sec. **2.** $\pi/6, =11/21$. $120/\pi, =38\frac{2}{11}$ feet. **5.** 1 : 2. **6.** (i) $18\frac{3}{4}$ lbwt. (ii) $187\frac{1}{2}$ lbwt. **7.** 80 ft./sec. **8.** 24 ft./sec. **9.** 1467·9. **10.** 7 ft./sec. $1\frac{2}{3}$ radians per second. **12.** 55561 foot-pounds. 44 turns.

XXI. (Page 290.)

1. (i) $5\frac{5}{8}$ feet. (ii) A pressure of $6\frac{1}{8}$ feet of water, $=383$ lbwt. per square foot. (iii) A pressure of 2 feet of water $=125$ lbwt. per square foot. (iv) A pressure of $\frac{23}{24}$ foot of water, $=59·9$ lbwt. per square foot. (v) 73°. **2.** 2 revolutions per second. **3.** $1\frac{7}{48}$ feet. **5.** 60·7 lbwt.

XXII. (Pages 317–318.)

1. 1/10. **2.** 1/64 of atmospheric pressure. **3.** $8\frac{8}{9}$ inches high. **4.** A pressure of 537·6 millimetres. **5.** 2·7 inches of mercury. **6.** Diminution of pressure is $\{1-(10/11)^4\}$ of the original pressure. **7.** 160. **8.** $6860\dot{·}5$ lbwt. **9.** 138·9 lbwt. **12.** 2·15 feet. **13.** 1097. **14.** 94·7 feet.

ANSWERS—MISCELLANEOUS EXAMPLES.

(Pages 319–324.)

1. 1·85 and 1·15 inches respectively. **2.** 2 inches. **3.** See Art. 61. The values of g at different latitudes would be proportional to the readings of the gauge. **4.** The total pressure on the side of the cylinder containing water would be s times the total pressure on the side of the cylinder containing mercury. **5.** 285 grammes weight. **6.** On the upper horizontal face 2200; on the lower horizontal face 2275; on each of the vertical faces 2237·5. **7.** 603000 grammes weight. **8.** Density of olive oil relative to water is ·92, of turpentine is ·68. **10.** 164·9 lbwt. **11.** $aW/(A-a)$. **12.** Let h and k denote the lengths of the perpendiculars from A and B respectively on the line of intersection of the plane of the triangle and the surface of the liquid; let $D_1, D_2, D_3, \ldots D_r, \ldots$ be points in CB such that AD_1, AD_2, $AD_3, \ldots AD_r, \ldots$ are the lines required; then D_r is determined from the equation $CD_r/CB=\{\sqrt{h^2+4rk(h+k)/n}-h\}/2k$. **13.** In the diagonal which is vertical at a depth $55c/36$ below the surface, c being the depth of the highest point of the square.

14. $(3h^2+hk+3k^2)/6(h+k)$. **16.** $h/2$. **17.** $3a\pi/16$. **18.** (i) 77·44 lbwt. (ii) 12·73 lbwt. **19.** (i) $84\frac{6}{7}$ lbwt. (ii) $198\sqrt{10}/7=89{\cdot}45$ lbwt. **20.** The horizontal and vertical components are $wh^3/3$ and $\pi wh^3/6$ respectively, the resultant pressure being the square root of the sum of the squares of these expressions. **21.** (i) $\pi r^3hw/\sqrt{r^2+h^2}$. (ii) Resultant vertical pressure on the curved surface is $\pi r^2hw(4r^2+h^2)/3(r^2+h^2)$, and resultant horizontal pressure is $\pi r^3h^2w/(r^2+h^2)$. **23.** Gold, 97 ounces; silver, 882 ounces. **24.** 5. **25.** 4 centimetres very nearly. **26.** If V is the whole volume of the solid the results are: (i) $3V/(a+b+c)$. (ii) $V(bc+ca+ab)/3abc$. **27.** 1·2. **28.** ·9. **29.** 2·5. **30.** The specific gravity of A is $2s_2s_3/(s_2+s_3)$, and the specific gravities of B and C are given by similar expressions. **31.** 2·3. **32.** 3 inches. **34.** 1·32. **35.** At a depth $104a/113$, where a is the edge of the cube. **37.** As 45 : 32. **38.** 60°. **40.** $\pi^2/4, =2{\cdot}5$ feet nearly. **41.** As 15 : 17. A pressure of 73·9 centimetres of mercury. **42.** 72 centimetres. **43.** $3h/2$, where h is the height of the water barometer. **44.** 150 lbwt. **45.** 20000 lbs. **46.** 33 feet. **47.** A pressure of $\frac{64}{17}$ inches of mercury very nearly. **48.** As $m/(b-h)(l-h) : m'/(b-h')(l-h')$. **49.** 33.

ANSWERS TO EXAMINATION PAPERS.

I. Science and Art—Fluids, 1892. (Pages 325–326.)

1. See Art. 49, page 73. **2.** See Ex. 18, page 105. **3.** 1000 foot-poundals. $14\frac{7}{12}$ poundals. **4.** See Ex. 10, page 284. **5.** See Art. 74, page 119. **6.** See Ex. at end of Art. 76, page 124. **7.** See Art. 137, page 228. $-459{\cdot}4°$ F. **8.** See Ex. 2, page 216. **9.** See Ex. 1, page 206. **10.** See Art. 162, page 264. **11.** See Ex. on page 275. **12.** See Arts. 175 and 176.

II. Science and Art—Fluids, 1893. (Pages 326–327.)

1. See Art. 50, page 74. **2.** The pressure at the point is 6000 grains weight per square inch. The horizontal and vertical components of this pressure are $3000\sqrt{3}$ and 3000 grains respectively. **3.** $4\frac{2}{7}$ radians per second. 5691·84 foot-poundals. **4.** Taking a cubic foot of water to weigh 62·5 lbs., and π to be 22/7, the *total* pressure is $9428\frac{4}{7}$ lbwt., and the *resultant* pressure is $2095\frac{5}{21}$ lbwt. **5.** See Art. 101, page 177. The diameter is not less than $\sqrt{3/2}, =1{\cdot}225$ times the height. **6.** 500. **7.** See Art. 138, page 229. As 733 : 229320. **8.** See Ex. 1, page 290. **9.** The point of the stem which is 1·6 inches above A. **10.** See Art. 141, page 234. **11.** 150 cubic feet. **12.** See Art. 178, page 297.

III. Science and Art—Fluids (Honours). (Pages 328–329.)

A.

1. See Ex. 24, page 29. **2.** (*a*) $4\sqrt{3}$ lbwt. (*b*) $6\sqrt{3}$ lbwt. The tension at the instant the mass is vertically below A is $(36 - 12\sqrt{3})$ lbwt. **3.** $\frac{88}{78}$. **4.** Compare with Ex. 37, page 189. **5.** The radius of the base must not be less than $2\sqrt{2/5}$ times the length of the axis. **6.** See Ex. 15, page 220. **7.** See Ex. 23, page 29.

B.

1. As $AC^2 : AB^2$. **2.** The forces are as 4 : 9, and the accelerations produced by them are as 4 : 21. **3.** 24 oz. **4.** See Ex. 2, page 179.

IV. Glasgow University Preliminary Examination. Dynamics. (Page 329.)

1. The pressure on the base is 456·25 lbwt., and on one of the sides is 129·6875 lbwt. **2.** 50 cubic centimetres 80 grammes. **3.** See Art. 118, page 198. **4.** See Ex. 1, page 190.

V. Glasgow University M.A. Degree Examination. Natural Philosophy. (Page 330.)

1. See Arts. 4 and 5. **2.** See note to Art. 56 below. For second part of the question see Art. 67, p. 101. **3.** See Arts. 78, 88, and 97. **4.** 11151 cubic feet.

VI. Edinburgh University Preliminary Examination. Dynamics. (Page 330.)

1. $1\frac{1}{3}$. **2.** 5000 lbwt.

VII. Edinburgh University M.A. Degree Examination. Natural Philosophy. (Page 331.)

1. 7 lbs. 7/8. **2.** The pressure on the bottom is 10000 lbwt., and on one of the sides is 12500 lbwt. **3.** 217 lbwt. nearly. **4.** The volume is increased in the ratio of 3730 to 3553.

NOTE TO ART. 56.

The following is a proof from the fundamental property of a fluid at rest that the pressure at a point is the same in all directions.

First suppose the fluid to be *without weight*.

Consider the equilibrium of a *very small* right prism of fluid. This fluid

is at rest under the action of the resultant pressures on its ends, and the resultant pressures on its faces. The resultant pressures on the ends act parallel to the axis of the prism, and the resultant pressures on the faces act perpendicular to the axis. It follows that the latter pressures must be in equilibrium.

Now since the faces of the prism are very small, we may suppose that the pressure on each face is of uniform intensity. Hence the resultant pressure on any one of the faces will be a force acting perpendicular to the face through its centre of gravity, and therefore through the centre of gravity of the prism, the point O say. The resultant pressures on the faces will therefore be forces acting through O in the plane perpendicular to the axis, each force being perpendicular to a side of the triangle in which this plane cuts the prism.

Let p_1, p_2, p_3 denote the intensities of pressure on the faces, and A, B, C the areas of the faces respectively. Then p_1A, p_2B, p_3C are the resultant pressures on the faces. Now since these forces are in equilibrium, and since they act at the point O perpendicular to the sides of the triangle in which the plane through O perpendicular to the axis cuts the prism, it follows from the *Triangle of Forces* that they are proportional to the sides to which they are respectively perpendicular. But these sides are proportional to the areas of the faces in which they lie respectively. Hence, finally, p_1A, p_2B, p_3C are proportional to A, B, C, or

$$p_1A/A = p_2B/B = p_3C/C,$$

that is,

$$p_1 = p_2 = p_3.$$

Hence the intensities of pressure at the point O in the directions perpendicular to the faces of the prism are equal. This proves the proposition, for, keeping the direction of one of the faces fixed, the directions of the other two faces may be taken perpendicular to any given directions.

When the fluid *has weight*, in considering the equilibrium of a very small prism of the fluid we may leave the weight of the fluid prism out of account in comparison with the pressures on the faces of the prism. For the weight is found by multiplying the weight of unit volume, which is a finite quantity, by the volume of the prism, and the pressure on a face is found by multiplying the area of the face by the average intensity of pressure on the face, this latter being a finite quantity. Now in comparison with the area of a face the volume of the prism is an infinitesimal quantity, for the volume of the prism is found by multiplying the area of a face by a length, which, since the prism is of very small dimensions, must be very small. Hence in considering the equilibrium of a very small prism of a heavy fluid the weight of the fluid in the prism may be left out of account.

Hence it follows that the intensity of pressure at a point of a fluid, whether heavy or weightless, is the same in all directions.

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